A free boundary problem driven by the biharmonic operator

A FREE BOUNDARY PROBLEM DRIVEN BY THE BIHARMONIC OPERATOR

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Abstract. In this paper we consider the minimization of the functional

$$J[u] := \int_{\Omega} |\Delta u|^2 + \chi_{\{u>0\}}$$

in the admissible class of functions

$$\mathcal{A} := \left\{ u \in W^{2,2}(\Omega) \text{ s.t. } u - u_0 \in W^{1,2}_0(\Omega) \right\}.$$

Here, Ω is a smooth and bounded domain of $\mathbb{R}^n$ and $u_0 \in W^{2,2}(\Omega)$ is a given function defining the Navier type boundary condition.

When $n = 2$, the functional J can be interpreted as a sum of the linearized Willmore energy of the graph of u and the area of $\{u > 0\}$ on the xy plane.

The regularity of u and that of the free boundary $\partial\{u > 0\}$ are very complicated problems. The most intriguing part of this is to study the structure of $\partial\{u > 0\}$ near singular points, where $\nabla u = 0$ (of course, at the nonsingular free boundary points where $\nabla u \neq 0$ the free boundary is locally C^1 smooth).

The scale invariance of the problem suggests that, at the singular points of the free boundary, quadratic growth of u is expected. We prove that u is quadratically nondegenerate at the singular free boundary points using a refinement of Whitney's cube decomposition, which applies, if, for instance, the set $\{u > 0\}$ is a John domain.

The optimal growth is linked with the approximate symmetries of the free boundary. More precisely, if at small scales the free boundary can be approximated by zero level sets of a quadratic degree two homogeneous polynomial, then we say that $\partial\{u > 0\}$ is rank-2 flat.

Using a dichotomy method for nonlinear free boundary problems, we also show that, at the free boundary points $x \in \Omega$ where $\nabla u(x) = 0$, the free boundary is either well approximated by zero sets of quadratic polynomials, i.e. $\partial\{u > 0\}$ is rank-2 flat, or u has quadratic growth.

More can be said if $n = 2$, in which case we obtain a monotonicity formula and show that, at the singular points of the free boundary where the free boundary is not well approximated by level sets of quadratic polynomials, the blow-up of the minimizer is a homogeneous function of degree two.

In particular, if $n = 2$ and $\{u > 0\}$ is a John domain, then we get that the blow-up of the free boundary is a cone, and in the one-phase case it follows that $\partial\{u > 0\}$ possesses a tangent line in the measure theoretic sense.

Differently from the classical free boundary problems driven by the Laplacian operator, the one-phase minimizers present structural differences with respect to the minimizers, and one notion is not included into the other. In addition, one-phase minimizers arise from the combination of a volume type free boundary problem and an obstacle type problem, hence their growth condition is influenced in a non-standard way by these two ingredients.

Contents

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1. Introduction

1.1. **Mathematical framework and motivations.** In this paper we consider the problem of minimizing the functional

$$J[u] := \int_\Omega |\Delta u|^2 + \chi_{\{u>0\}} \tag{1.1}$$

over the admissible class of functions

$$\mathcal{A} := \left\{ u \in W^{2,2}(\Omega) \text{ s.t. } u - u_0 \in W_0^{1,2}(\Omega) \right\}. \tag{1.2}$$

Here, Ω is a smooth and bounded domain of $\mathbb{R}^n$ and $u_0 \in W^{2,2}(\Omega)$ is a given function defining the Navier type boundary condition (see e.g. the "hinged problem" on the right hand side of Figure 1(a) and on page 84 of [Swe09], or Figure 1.5 on page 6 of [Gan17], or the monograph [GGS10] for additional information of this condition, which can be interpreted as a weak form of two boundary conditions: $u = u_0$ along $\partial\Omega$ and $\Delta u = 0$ along $\partial\Omega \cap \{u \neq 0\}$).

The functional in (1.1) is clearly related to the biharmonic operator, which provides classical models for rigidity problems with concrete applications, for instance, in the construction of suspension bridges, see e.g. [MW87] and the references therein. Other classical applications of the biharmonic operator arise in the study of steady state incompressible fluid flows at small Reynolds numbers under the Stokes flow approximation assumption, see e.g. formula (1) in [MZ16] and the references therein. In our setting, we will provide a simple mechanical interpretation of the model in Section 6.

Moreover, the functional in (1.1) provides a linearized model for the Willmore problem which asks to find an immersion/embedding M in $\mathbb{R}^3$ that minimizes the Willmore energy

$$W(M) = \int_M H^2\, dA,$$

where H denotes the mean curvature. The linearization of this energy density gives

$$H^2\, dA = \frac{1}{4}(\Delta u)^2\, dx\, dy + \text{lower order terms}.$$

In this context our problem can be regarded as a free boundary problem for the linearized Willmore energy, where the surface M has a flat part on the xy plane.

We also refer to the very recent work in [DPR18] for a problem related to the minimization of the Willmore energy functional with prescribed boundary, boundary Gauss map and area. See also the recent contributions in [Miu16, Miu17] for the one-dimensional analysis of the global properties of the solutions of free boundary problems involving the curvature of a curve.

In the setting of (1.1), an additional motivation for us comes from the study of the degenerate/unstable obstacle problem, see [Caf80, MW07]. Indeed, we will see in Corollary 4.2 that u is globally almost subharmonic in Ω, i.e. there exists a constant $\hat{C} > 0$ such that $\Delta u \geqslant -\hat{C}$. Therefore the function $\Delta u := f$ is bounded from below. Accordingly, we can relate our problem to an obstacle problem with unknown right hand side, namely determine u and $f \geqslant -\hat{C}$ such that

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\{u > 0\}, \\ f = 1 & \text{on } \partial\{u > 0\}. \end{cases} \tag{1.3}$$

The principal difference from the classical obstacle problem is that f may change sign in Ω and degenerate on the free boundary points, since the last condition in (1.3) is satisfied in a generalized sense: for this reason, it does not follow from the classical obstacle problem theory that u is quadratically nondegenerate.

Another motivation for the problem in (1.1) comes from the limit as $\varepsilon \to 0$ of the singularly perturbed bi-Laplacian equation

$$\Delta^2 u^\varepsilon = -\frac{1}{\varepsilon}\beta\left(\frac{u^\varepsilon}{\varepsilon}\right), \tag{1.4}$$

where β is a compactly supported nonnegative function with finite total mass. Equation (1.4) can be seen as the biharmonic counterpart of classical combustion models, see e.g. [Pet02].

1.2. **Comparison with the existing literature.** Free boundary problems are of course a classical topic of investigation, nevertheless only few results are available concerning the case of equations of order higher than two, and there seems to be no investigation at all for the free boundary problem in (1.1).

Other types of free boundary problems for higher order operators have been considered in [Maw14]. Moreover, obstacle problems involving biharmonic operators have been studied in [Fre73, CF79, CFT81, CFT82, AV00, PL08, NO15, NO16, Ale16], but till now we are not aware of any previous investigation of free boundary problems dealing with higher order operators combined with "bulk" volume terms as in (1.1) here.

Of course, one of the striking differences in our framework, as opposed to the case of the Alt-Caffarelli functional (see [AC81])

$$J_{\mathrm{AC}}[u] := \int_\Omega |\nabla u|^2 + \chi_{\{u>0\}},$$

is the lack of Maximum Principle and Harnack inequality for higher order operators. This, in our setting, reflects to the fact that the set $\{u < 0\}$ may be nonempty, even under the boundary condition $u_0 \geqslant 0$. This is one of the peculiars of the situations involving the bi-Laplacian and it makes the mathematical treatment of the problem extremely difficult (and this is likely to be the reason for which there are not many results in the direction of free boundary regularity in the framework that we consider here).

Thus, the main difficulties in our setting, in comparison with the existing literature, follow from the fact that major elliptic methods based on Maximum Principle, Harnack inequality and propagation of ellipticity cannot be applied. Moreover, many classical tools, such as domain variations, have not been fully analyzed yet and, in any case, cannot provide consequences which are as strong as in the classical framework. For instance, the main result that we obtain by domain variation (given in details in the forthcoming Lemma 4.3) is that, for any $\phi = (\phi^1, \dots, \phi^n) \in C_0^\infty(\Omega)$,

$$2\int_\Omega \Delta u(x) \sum_{m=1}^n \Big(2\nabla u_m(x)\cdot\nabla\phi^m(x) + u_m(x)\Delta\phi^m(x)\Big)\,dx = \int_\Omega \Big(|\Delta u(x)|^2 + \chi_{\{u>0\}}(x)\Big)\mathrm{div}\phi(x)\,dx. \tag{1.5}$$

Then, in the classical literature, the standard argument leading to the monotonicity formula for the Alt-Caffarelli problem would be to choose ϕ of a particular form, see [Wei98]. More precisely, for $\varepsilon > 0$, the classical idea would be to consider

$$\eta(x) := \begin{cases} 1 & \text{if } x \in B_r(x_0), \\ \dfrac{r+\varepsilon-|x-x_0|}{\varepsilon} & \text{if } x \in B_{r+\varepsilon}(x_0) \setminus B_r(x_0), \\ 0 & \text{otherwise,} \end{cases}$$

where $x_0 \in \partial\{u > 0\}$, and take $\phi(x) := x\eta(x)$ in identity (1.5). Note that

$$\nabla\phi(x) = \begin{cases} \mathbb{I} & \text{if } x \in B_r(x_0), \\ \mathbb{I}\eta - \dfrac{1}{\varepsilon}\dfrac{(x-x_0)\otimes(x-x_0)}{|x-x_0|} & \text{if } x \in B_{r+\varepsilon}(x_0) \setminus B_r(x_0), \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{I} \in \mathrm{Mat}_{n\times n}$ is the identity matrix. However, in our case, identity (1.5) contains the term $\Delta\phi$ which is not defined on the boundary of the ring $B_{r+\varepsilon}(x_0)\setminus B_r(x_0)$ and this creates an important conceptual difficulty. Thus, to overcome this issue, one needs to perform a series of ad-hoc integration by parts. This strategy has however to confront with the possible generation of third order derivatives of the minimizers, which also cannot be controlled, therefore these terms need to be suitably smoothened and simplified via appropriate cancellations.

In this setting, the lack of monotonicity formulas can also be seen as a counterpart of a lack of Pohozhaev type inequalities, and our approach bypasses this kind of difficulty.

As a matter of fact, we will establish a new monotonicity formula in dimension 2 which will lead to the forthcoming Theorem 1.13.

In addition, differently from the harmonic case, there are no estimates available in the literature for the biharmonic measure, and this makes the free boundary analysis significantly more complicated. We will overcome these difficulties by the forthcoming Theorem 1.10.

Moreover, in terms of barrier and test functions, an additional difficulty of the biharmonic setting is given by the fact that the function $\max\{u, v\}$ is not an admissible competitor, having possibly infinite energy, so we cannot consider the maximal and minimal solutions.

The analysis of nondegeneracy and optimal regularity of minimizers and of their free boundary is also a novel ingredient with respect to the classical literature, and nothing seemed to be known before about these important questions.

1.3. **Main results.** In what follows, we will denote by $\{u > 0\}$ the positivity set of u and by $\partial\{u > 0\}$ its free boundary. The main results of this paper are the following:

- If $z \in \partial\{u > 0\}$ and $\nabla u(z) = 0$, then either $\partial\{u > 0\}$ can be approximated by the zero level sets of a quadratic homogeneous polynomial of degree two, or u has quadratic growth at z.
- If $n = 2$, there exists a monotonicity formula and we can classify the homogeneous one-phase solutions of degree two.
- We also provide various sufficient conditions for strong nondegeneracy in terms of a suitable refinement of Whitney's cube decomposition (c-covering). For instance, we show that if $\{u > 0\}$ is a John domain (see the definition in Subsection 8.2), then $\partial\{u > 0\}$ possesses a measure theoretic tangent line.

In further details, the first regularity result that we establish is a BMO estimate on the Laplacian of the minimizers. Namely, we prove that:

Theorem 1.1. *Let u be a minimizer of the functional J defined in (1.1). Then, we have that $\Delta u \in BMO_{\mathrm{loc}}(\Omega)$.*

We also introduce a notion of one-phase minimizer, in the following setting:

Definition 1.2. *We say that u is a one-phase minimizer of J if it minimizes the functional J in (1.1) among the nonnegative admissible functions $\{u \in \mathcal{A}$ s.t. $u \geqslant 0$ in $\Omega\}$, $\mathcal{A}$ being as in (1.2).*

Interestingly, one-phase minimizers, as given in Definition 1.2, arise from a combination of a biharmonic free boundary problem and an obstacle problem. We also observe that, in general, minimizers of J which happen to be nonnegative do not naturally develop open regions in which the minimizer vanishes (see Proposition B.1 for a concrete result), while one-phase minimizers do (hence, the notion of minimizers that are nonnegative and the notions of one-phase minimizers are structurally very different in this framework, due to the lack of maximum principle).

We stress that one-phase minimizers, as given in Definition 1.2, are not necessarily minimizers over $\mathcal{A}$. This fact produces significant differences with respect to the classical case of free boundary problems driven by the Laplacian, and requires some non-standard techniques to overcome the lack of structure provided, in the classical case, by super-harmonic functions.

Given the higher order structure of the biharmonic functional, the minimizers satisfy a free boundary condition which is richer, and more complicated, than in the harmonic case. To express it in a general form, suppose that the free boundary (locally) separates two regions, say $\Omega^{(1)}$ and $\Omega^{(2)}$, of the domain Ω, with $\partial\Omega^{(1)} = \partial\Omega^{(2)} = \partial\{u > 0\}$: in this case, the minimizer u can be seen as the result of the junction of two functions, say $u^{(1)}$ and $u^{(2)}$, from each side of the free boundary, with $u^{(1)}$ and $u^{(2)}$ not changing sign. In this notation, for $i \in \{1, 2\}$, we set

$$\lambda^{(i)} := \begin{cases} 1 & \text{if } u^{(i)} > 0, \\ 0 & \text{if } u^{(i)} \leqslant 0. \end{cases} \tag{1.6}$$

Then, we have the following result describing the free boundary condition in this framework:

Theorem 1.3. *Let u be either a minimizer or a continuous one-phase minimizer of the functional J defined in (1.1). Assume that*

$$\partial\{u > 0\} \textit{ is of class } C^1. \tag{1.7}$$

Then, for any $\phi = (\phi^1, \dots, \phi^n) \in C_0^\infty(\Omega)$,

$$\begin{aligned} &\int_{\partial\{u>0\}} \left(\left(|\Delta u^{(1)}|^2 + \lambda^{(1)}\right)\phi\cdot\nu - 2\sum_{m=1}^{n} \left(\phi^m \left(\Delta u^{(1)} \nabla u^{(1)}_m - u^{(1)}_m \nabla \Delta u^{(1)}\right)\cdot\nu + \Delta u^{(1)} u^{(1)}_m \nabla\phi^m\cdot\nu \right)\right) \\ &\quad = \int_{\partial\{u>0\}} \left(\left(|\Delta u^{(2)}|^2 + \lambda^{(2)}\right)\phi\cdot\nu - 2\sum_{m=1}^{n} \left(\phi^m \left(\Delta u^{(2)} \nabla u^{(2)}_m - u^{(2)}_m \nabla \Delta u^{(2)}\right)\cdot\nu + \Delta u^{(2)} u^{(2)}_m \nabla\phi^m\cdot\nu \right)\right), \end{aligned} \tag{1.8}$$

where ν is the exterior normal to $\Omega^{(1)}$.

Furthermore, if $u \in C^3\left(\overline{\Omega^{(1)}}\right) \cap C^3\left(\overline{\Omega^{(2)}}\right)$, we have that

$$\begin{cases} \qquad \Delta u^{(1)} u^{(1)}_m = \Delta u^{(2)} u^{(2)}_m \\ \\ \textit{and} \quad \left(|\Delta u^{(1)}|^2 + \lambda^{(1)}\right)\nu_m - 2\left(\Delta u^{(1)} \nabla u^{(1)}_m - u^{(1)}_m \nabla\Delta u^{(1)}\right)\cdot\nu \\ \qquad\qquad = \left(|\Delta u^{(2)}|^2 + \lambda^{(2)}\right)\nu_m - 2\left(\Delta u^{(2)} \nabla u^{(2)}_m - u^{(2)}_m \nabla\Delta u^{(2)}\right)\cdot\nu, \end{cases} \tag{1.9}$$

for any $m \in \{1, \dots, n\}$, on $\partial\{u > 0\}$.

Concrete examples of this free boundary condition, together with some applications from mechanics, will be also presented in Sections 5 and 6.

As already discussed in Subsection 1.2, one of the principal features of the problem that we consider in the present work is that it does not share the standard properties of its "sibling" Alt-Caffarelli problem [AC81], such as non degeneracy, linear growth, etc. Moreover, the existing techniques fail because of the involvement of higher order derivatives.

However, the scale invariance of the functional suggests that the optimal regularity of u must be $C^{1,1}$. This is also supported by the computations that we have for the one-dimensional case (see the forthcoming Remark 4.4 and the explicit examples in Section 5).

Now, to study the free boundary points, it is useful to distinguish between regular and singular points. Related to this, suppose that $x \in \partial\{u > 0\}$, then there are two possible cases:

- $\nabla u(x) \neq 0$, then $\partial\{u > 0\}$ is C^1 near x.
- $\nabla u(x) = 0$, then we expect u to grow quadratically and the free boundary may have self-intersections.

To analyze these situations, we introduce the following setting:

Definition 1.4. *If $x \in \partial\{u > 0\}$ and $\nabla u(x) = 0$, then we say that x is a singular free boundary point. The set of singular points is denoted by $\partial_{sing}\{u > 0\}$.*

Clearly the singular points are the most interesting points of the free boundary to study. In order to overcome all the difficulties mentioned in Subsection 1.2 and study the regularity of u and that of the free boundary $\partial\{u > 0\}$, we employ a dichotomy argument which was introduced in [DK18]. The idea is to exploit a suitable notion of "flatness" and distinguish between points where the free boundary is flat and points where it is nonflat, according to this new notion.

To this aim, we let

$$\mathrm{HD}(A, B) := \max\left\{\sup_{a\in A} \mathrm{dist}(a, B),\ \sup_{b\in B} \mathrm{dist}(b, A)\right\} \tag{1.10}$$

be the Hausdorff distance of two sets A, $B \subset \mathbb{R}^n$.

We also let P_2 be the set of all homogeneous polynomials of degree two, i.e.

$$P_2 := \left\{p(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j,\ \text{for any } x \in \mathbb{R}^n,\ \text{with } \|p\|_{L^\infty(B_1)} = 1\right\}, \tag{1.11}$$

where a_{ij} is a symmetric $n \times n$ matrix. Moreover, given $p \in P_2$ and $x_0 \in \mathbb{R}^n$, we set $p_{x_0}(x) := p(x - x_0)$ and

$$S(p, x_0) := \{x \in \mathbb{R}^n : p_{x_0}(x) = 0\}. \tag{1.12}$$

We observe that the set $S(p, x_0)$ defined in (1.12) is a cone with vertex at x_0, i.e. if $x \in S(p, x_0)$ then, for every $t > 0$, it holds that $x_0 + t(x - x_0) \in S(p, x_0)$.

With this notation, we set:

Definition 1.5. *Let $\delta > 0$, $R > 0$ and $x_0 \in \partial\{u > 0\}$. We say that $\partial\{u > 0\}$ is (δ, R)-rank-2 flat at x_0 if, for every $r \in (0, R]$, there exists $p \in P_2$ such that*

$$\mathrm{HD}\Big(\partial\{u > 0\} \cap B_r(x_0), S(p, x_0) \cap B_r(x_0)\Big) < \delta\, r.$$

Now, given $r > 0$, $x_0 \in \partial\{u > 0\}$ and $p \in P_2$, we let

$$h_{\min}(r, x_0, p) := \mathrm{HD}\Big(\partial\{u > 0\} \cap B_r(x_0), S(p, x_0) \cap B_r(x_0)\Big). \tag{1.13}$$

Then, we define the rank-2 flatness at level $r > 0$ of $\partial\{u > 0\}$ at x_0 as follows. We set

$$h(r, x_0) := \inf_{p\in P_2} h_{\min}(r, x_0, p) \tag{1.14}$$

and we introduce the following notation:

Definition 1.6. *Let $\delta > 0$, $r > 0$ and $x_0 \in \partial\{u > 0\}$. We say that $\partial\{u > 0\}$ is δ-rank-2 flat at level r at x_0 if $h(r, x_0) < \delta r$.*

In view of Definitions 1.5 and 1.6, we can say that $\partial\{u > 0\}$ is (δ, R)-rank-2 flat at $x_0 \in \partial\{u > 0\}$ if and only if, for every $r \in (0, R]$, it is δ-rank-2 flat at level r at x_0.

With this, we can now state the following result concerning the quadratic growth of u at "nonflat" points of the free boundary.

Theorem 1.7. *Let $n \geqslant 2$ and u be a minimizer of the functional J defined in (1.1). Let $D \Subset \Omega$, $\delta > 0$ and let $x_0 \in \partial\{u > 0\} \cap D$ such that $|\nabla u(x_0)| = 0$ and $\partial\{u > 0\}$ is not δ-rank-2 flat at x_0 at any level $r > 0$.*

Then, u has at most quadratic growth at x_0, bounded from above in dependence on δ.

We also study nondegeneracy properties of the minimizers. First of all, setting as usual $u^+(x) := \max\{u(x), 0\}$, we provide a weak form of nondegeneracy, investigating the validity of statements of this form:

$$\text{if } B \subset \{u > 0\} \text{ is a ball touching } \partial\{u > 0\}, \text{ then } \sup_B u^+ \geqslant C[\operatorname{diam}(B)]^2 \tag{1.15}$$

for some universal constant $C > 0$. We consider this as a weak form of nondegeneracy as opposed to the one in which B is centered at free boundary points, which we call strong nondegeneracy.

We establish that (1.15) is satisfied, and, more generally, that the positive density of the positivity set is sufficient to ensure at least quadratic growth from the free boundary. The precise result that we obtain is the following:

Theorem 1.8. *Let u be a minimizer of the functional J defined in (1.1). Then:*

1° *If $x_0 \in \partial\{u > 0\}$ and*

$$\liminf_{r\to 0} \frac{\big|B_r(x_0) \cap \{u > 0\}\big|}{|B_r|} \geqslant \theta_* \tag{1.16}$$

for some $\theta_* > 0$, then

$$\sup_{B_r(x_0)} u \geqslant \bar{c}r^2$$

for some positive constant $\bar{c} > 0$ depending on θ_*.

2° *If $x_0 \in \{u > 0\}$ and $\rho := \operatorname{dist}(x_0, \partial\{u > 0\})$, then there exists a positive constant $\bar{c}$ depending only on n and $\|u_0\|_{W^{2,2}}$ such that*

$$\sup_{B_\rho(x_0)} u^+ \geqslant \bar{c}\rho^2.$$

We observe that the claim in **2°** is exactly the statement in (1.15).

Sufficient conditions for the density estimate in (1.16) to hold will be discussed in Subsection 8.2, where we also recall and compare the notions of weak c-covering condition and Whitney's covering. In addition, in Subsection 8.3 we will relate the nondegeneracy properties with a fine analysis of the biharmonic measure, which in turn produces some regularity results on the free boundary.

It is also convenient to consider "vanishing" free boundary points, in the following sense:

Definition 1.9. *Let u be a minimizer of the functional J defined in (1.1) and let $x_0 \in \partial\{u > 0\} \cap B_1$. We say that $\partial\{u > 0\}$ is vanishing rank-2 flat at x_0 if there exist sequences $\delta_k \to 0$ and $r_k \to 0$ such that*

$$h(r_k, x_0) \leqslant \delta_k r_k, \tag{1.17}$$

where h is defined in (1.14).

Notice, in particular, that condition (1.17) is equivalent to

$$\lim_{k\to+\infty} \frac{h(r_k, x_0)}{r_k} = 0,$$

and this justifies the name of "vanishing" in Definition 1.9.

Then, we have:

Theorem 1.10. *Let u be a minimizer of the functional J defined in (1.1). Then:*

1° *The set of vanishing rank-2 flat points of the free boundary has zero measure in Ω.*

2° *If $D \Subset \Omega$ and there exists a constant $\bar{c} > 0$ such that*

$$\liminf_{r\to 0} \frac{\sup_{B_r(x)} |u|}{r^2} \geqslant \bar{c} \tag{1.18}$$

for every $x \in \partial\{u > 0\} \cap \overline{D}$, then $\partial\{u > 0\}$ has zero measure, and for any $\delta > 0$, the set of free boundary points that are not δ-rank-2 flat has finite $(n-2)$-dimensional Hausdorff measure.

In general, we can restate the previous results in a dichotomy form: roughly speaking, the free boundary in the vicinity of singular points is either "flat" with respect to the level sets of homogeneous polynomial of degree two, being "close" to the level sets of quadratic polynomials, or "non-flat" and in this case the growth from the free boundary is quadratic. To formalize these notions, we decompose the class P_2 introduced in (1.11) as

$$P_2 = \bigcup_{i=1}^{n} P_2^i,$$

where

$$P_2^i := \left\{ p \in P_2 : \mathrm{Rank}(D^2 p) = i \right\}.$$

As we will see, in our setting, the above notion will play a useful role since if $x_0 \in \partial\{u > 0\}$, with $|\nabla u(x_0)| = 0$, and $\partial\{u > 0\}$ is rank-2 flat at x_0, then there exists $p \in P_2$ such that the blow-up of $\partial\{u > 0\}$ at x_0 is the zero set of p. We separate out some interesting cases:

- If $\mathrm{Rank}(D^2p) = n$ and $D^2p \geqslant 0$ then the free boundary is a singleton.
- If $\mathrm{Rank}(D^2p) = 1$ then the free boundary is a hyperplane in $\mathbb{R}^n$, i.e. a codimension 1 plane in $\mathbb{R}^n$ and after some rotation of coordinates we can write $p(x) = \alpha(x_1^+)^2$, where $\alpha \in \mathbb{R}$ is a normalizing constant.
- If $\mathrm{Rank}(D^2p) = n$ and D^2p has eigenvalues of opposite signs then the free boundary has self intersection. For instance, if $n = 2$ then $p(x) = \alpha(x_1^2 - x_2^2)$, where $\alpha \in \mathbb{R}$ is a normalizing constant.

Roughly speaking, in this setting the classes P_2^i detect the approximate symmetries of the free boundary at small scales.

To describe an appropriate flatness rate of the minimizers, we recall Definition 1.4 and we also define a suitable class, in the following way:

Definition 1.11. *Fix $r > 0$. We say that $u \in \mathcal{P}_r$ if:*

- *$u \in W^{2,2}(B_r)$ is a minimizer of J in (1.1) in B_r, among functions $v \in W^{2,2}(B_r)$, and $v - u \in W_0^{1,2}(B_r)$,*
- *and $0 \in \partial_{sing}\{u > 0\}$.*

If, in addition, given $\delta > 0$,

- *the free boundary is not (δ, r)-rank-2 flat at 0,*

then we say that $u \in \mathcal{P}_r(\delta)$.

Also, we let $\mathcal{F}$ be the set of vanishing rank-2 flat free boundary points with $|\nabla u| = 0$, and

$$\mathcal{N} := \big(\partial\{u > 0\} \setminus \mathcal{F}\big) \cap \{|\nabla u| = 0\}.$$

In this framework, the main result in the stratification setting reads as follows:

Theorem 1.12. *We have that*

- *for any $z \in \mathcal{F}$, there exist $r_k \to 0$ and $p \in P_2^i$, for some $i \in \{1, \dots, n\}$, such that*

$$\lim_{k \to +\infty} \mathrm{HD}\big((\partial E_k) \cap B_R, \{p = 0\} \cap B_R\big) = 0 \tag{1.19}$$

  for every fixed $R > 0$, where

$$E_k := \{x \in \mathbb{R}^n : z + r_k x \in \{u > 0\}\}.$$

  Furthermore, u^+ is strongly nondegenerate at z, namely

$$\sup_{B_r(z)} u^+ \geqslant c r^2,$$

  for some $c > 0$, as long as $B_r(z) \Subset \Omega$, with c possibly depending on $\mathrm{dist}(z, \partial\Omega)$;
- *for any $z \in \mathcal{N}$, there exists a constant $C_z > 0$ such that*

$$|u(x)| \leqslant C_z |x - z|^2 \tag{1.20}$$

  near z.

To analyze and classify the free boundary properties of the minimizers of J and their blow-up limits, it would be extremely desirable to have suitable monotonicity formulas. Differently from the classical case, in our setting no general result of this type is available in the literature. To overcome this difficulty, we focus on the two-dimensional case, for which we prove that:

Theorem 1.13. *Let $n = 2$ and $\tau > 0$ such that $B_\tau \Subset \Omega$. Let $u : \Omega \to \mathbb{R}$, with $0 \in \partial\{u > 0\}$ and $\nabla u(0) = 0$, be*

- *either: a minimizer of the functional J, with 0 not (δ, τ)-rank-2 flat in the sense of Definition 1.5,*
- *or: a one-phase minimizer of the functional J with $u \in C^{1,1}(\Omega)$, and such that $\partial\{u > 0\}$ has null Lebesgue measure.*

Then, there exists a function $E : (0, \tau) \to \mathbb{R}$, which is bounded, nondecreasing and such that, for any $\tau_2 > \tau_1 > 0$,

$$E(\tau_2) - E(\tau_1) = \int_{\tau_1}^{\tau_2} \left\{ \frac{1}{r^2} \int_{\partial B_r} \left[\left(\frac{u_{\theta r}}{r} - \frac{2u_\theta}{r^2} \right)^2 + \left(u_{rr} - \frac{3u_r}{r} + \frac{4u}{r^2} \right)^2 \right] \right\} dr. \tag{1.21}$$

The explicit value of the function E is given by

$$E(r) = \int_{\partial B_r} \left(\frac{\Delta u\, u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6uu_r}{r^4} + \frac{u_\theta u_{\theta r}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_\theta^2}{2r^5} \right) + \frac{1}{4r^2} \int_{B_r} \left(|\Delta u|^2 + \chi_{\{u>0\}} \right). \tag{1.22}$$

Furthermore, if E is constant in $(0, \tau)$, then u is a homogeneous function of degree two in B_τ.

Given $x_0 \in \partial\{u > 0\}$ we consider the blow-up sequence of u at x_0, defined as

$$u_k(x) := \frac{u(x_0 + \rho_k x)}{\rho_k^2}, \tag{1.23}$$

where $\rho_k \to 0$ as $k \to +\infty$.

In this setting, we can classify blow-up limits of minimizers in the plane, according to the following result:

Theorem 1.14. *Let $n = 2$. Let $B_r \Subset \Omega$. Let $x_0 \in \Omega$ and $u : \Omega \to \mathbb{R}$, with $x_0 \in \partial_{sing}\{u > 0\}$.*

Assume that either u is a minimizer of the functional J, with

$$\partial\{u > 0\} \text{ not } \delta\text{-rank-2 flat at } x_0 \text{ at any level,} \tag{1.24}$$

for some $\delta > 0$, or that u is a one-phase minimizer of the functional J with $u \in C^{1,1}(\Omega)$, and such that $\partial\{u > 0\}$ has null Lebesgue measure.

Then every blow-up limit of u at x_0 is either a homogeneous function of degree two, or it is identically zero.

One of the main issues in the free boundary analysis is that, even in the one-phase problem, the topological and measure theoretic boundaries of $\{u > 0\}$ may not coincide. On the other hand, following is a regularity result for the one-phase free boundary in the plane:

Theorem 1.15. *Let $n = 2$. Suppose that $B_1 \Subset \Omega$. Assume that u is a one-phase minimizer for J, that*

$$u \in C^{1,1}(B_1), \tag{1.25}$$

and that $\partial\{u > 0\}$ has null Lebesgue measure.

Suppose that $0 \in \partial_{sing}\{u > 0\}$. Assume also that, for every $\bar{x} \in \partial\{u > 0\} \cap B_1$,

$$\liminf_{\rho \to 0^+} \frac{\sup_{B_\rho(\bar{x})} u}{\rho^2} \geqslant c, \tag{1.26}$$

for some $c > 0$, for all $\rho \in (0, 1)$, and that

$$\limsup_{\rho \to 0} \frac{|B_\rho \cap \{u > 0\}|}{|B_\rho|} < 1. \tag{1.27}$$

Then there exists $r_0 > 0$ such that at every point $\bar{x}$ of $\partial\{u > 0\} \cap B_{r_0}$ the free boundary possesses a unique approximate tangent line in measure theoretic sense, namely if D is the symmetric difference of the sets $\{u > 0\}$ and a suitable rotation of $\{(x - \bar{x}) \cdot e_1 > 0\}$, we have that

$$\lim_{\rho \to 0^+} \frac{|B_\rho(\bar{x}) \cap D|}{|B_\rho(\bar{x})|} = 0.$$

1.4. **Organization of the paper.** The rest of the paper is organized as follows. Section 2 contains the main existence result. In Section 3 we provide the proof of the local BMO estimate for the Laplacian of the minimizers, as given by Theorem 1.1.

In Section 4 we present some structural properties of the minimizers which are based on the first variation of the functional J. As a consequence, we also obtain the free boundary condition and we prove Theorem 1.3.

In Section 5, we discuss some one-dimensional examples, and in Section 6 we provide a mechanical interpretation of the free boundary condition.

Section 7 contains a dichotomy argument which leads to the proof of Theorem 1.7.

Section 8 is devoted to nondegeneracy considerations and to the proof of Theorems 1.8 and 1.10.

In Section 9 we consider the stratification of the free boundary, reformulating some results obtained in Section 7, and, in particular, we prove Theorem 1.12.

Section 10 focuses on the monotonicity formula and contains the proof of Theorem 1.13.

In Section 11 we present an application of such a monotonicity formula, proving the homogeneity of the blow-up limits, and establishing Theorem 1.14.

Then, Section 12 focuses on explicit two-dimensional regularity and classification results and contains the proof of Theorem 1.15.

The paper ends with two appendices which collect some ancillary observations.

2. Existence of minimizers

The following result exploits the direct method of the calculus of variations to obtain the existence of the minimizers for our problem. Due to the presence of several technical aspects in the proof, we provide the argument in full details:

Lemma 2.1. *The functional in (1.1) attains a minimum over $\mathcal{A}$.*

Proof. Let $u_k \in \mathcal{A}$ be a minimizing sequence, namely

$$\lim_{k \to +\infty} J[u_k] = \inf_{v \in \mathcal{A}} J[v]. \tag{2.1}$$

For large k, we can suppose that

$$J[u_k] \leqslant J[u_0] + 1 \leqslant \int_\Omega (|\Delta u_0|^2 + 1) \leqslant C, \tag{2.2}$$

for some $C > 0$. Also, since $u_k \in \mathcal{A}$, we know from (1.2) that $u_k^* := u_k - u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Let also $v_k^* := \Delta u_k^* \in L^2(\Omega)$. In this way, we have that

$$\begin{cases} \Delta u_k^* = v_k^* & \text{in } \Omega, \\ u_k^* = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, by elliptic regularity (see Theorem 4 on page 317 of [Eva98]) we know that

$$\|u_k^*\|_{W^{2,2}(\Omega)} \leqslant C' \left(\|v_k^*\|_{L^2(\Omega)} + \|u_k^*\|_{L^2(\Omega)} \right), \tag{2.3}$$

for some $C' > 0$. Also (see Theorem 6 on page 306 of [Eva98]), one has that

$$\|u_k^*\|_{L^2(\Omega)} \leqslant C'' \|v_k^*\|_{L^2(\Omega)}, \tag{2.4}$$

for some $C'' > 0$. Therefore, in light of (2.3) and (2.4) we conclude that

$$\|u_k^*\|_{W^{2,2}(\Omega)} \leqslant C''' \|v_k^*\|_{L^2(\Omega)} = C''' \|\Delta u_k^*\|_{L^2(\Omega)}$$

for some $C''' > 0$. This and (2.2) imply that

$$\|u_k^*\|_{W^{2,2}(\Omega)} \leqslant C''''$$

for some $C'''' > 0$. Therefore, we can suppose, up to a subsequence, that

$$u_k^* \text{ converges to some } u^* \text{ weakly in } W^{2,2}(\Omega) \tag{2.5}$$

and then, by compact embedding,

$$u_k^* \text{ converges strongly to } u^* \text{ in } W^{1,2}(\Omega). \tag{2.6}$$

Since $u_k^* \in W_0^{1,2}(\Omega)$, this implies that also $u^* \in W_0^{1,2}(\Omega)$. As a consequence, recalling (1.2), we know that

$$u := u^* + u_0 \text{ belongs to } \mathcal{A}. \tag{2.7}$$

Furthermore, by (2.5), it holds that u_k converges to u weakly in $W^{2,2}(\Omega)$. In particular, u_k is bounded in $W^{2,2}(\Omega)$ and therefore, for any $i \in \{1, \dots, n\}$, it holds that $\partial_i^2 u_k$ is bounded in $L^2(\Omega)$. This yields that $\partial_i^2 u_k$ converges to some w_i weakly in $L^2(\Omega)$. This and

$$\text{the strong convergence of } u_k \text{ to } u \text{ in } W_0^{1,2}(\Omega) \subset L^2(\Omega) \tag{2.8}$$

(recall (2.6)) imply that, for any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega w_i \, \varphi = \lim_{k\to+\infty} \int_\Omega \partial_i^2 u_k \, \varphi = \lim_{k\to+\infty} \int_\Omega u_k \, \partial_i^2 \varphi = \int_\Omega u \, \partial_i^2 \varphi,$$

which shows that $w_i = \partial_i^2 u$.

Accordingly, we have that $\partial_i^2 u_k$ converges to $\partial_i^2 u$ weakly in $L^2(\Omega)$. Therefore, we have that

$$\begin{aligned} 0 \leqslant \lim_{k\to+\infty} \int_\Omega |\Delta(u_k - u)|^2 &= \lim_{k\to+\infty} \int_\Omega |\Delta u_k|^2 + \int_\Omega |\Delta u|^2 - 2 \int_\Omega \Delta u_k \Delta u \\ &= \lim_{k\to+\infty} \int_\Omega |\Delta u_k|^2 - \int_\Omega |\Delta u|^2. \end{aligned} \tag{2.9}$$

Now, up to a subsequence, recalling (2.8), we can suppose that u_k converges to u a.e. in Ω and therefore

$$\liminf_{k\to+\infty} \chi_{\{u_k>0\}} \geqslant \chi_{\{u>0\}}$$

a.e. in Ω. Consequently, by Fatou Lemma,

$$\liminf_{k\to+\infty} \int_\Omega \chi_{\{u_k>0\}} \geqslant \int_\Omega \chi_{\{u>0\}}.$$

Combining this with (2.9), we see that (2.1) gives that

$$J[u] \leqslant \liminf_{k\to+\infty} J[u_k] = \inf_{v\in\mathcal{A}} J[v].$$

This and (2.7) imply that u is the desired minimizer. □

By taking into account a nonnegative constraint in the minimizing sequence in the proof of Lemma 2.1, one also obtains an existence result for the one-phase problem.

3. BMO estimates and proof of Theorem 1.1

The goal of this section is to show that the minimizers of (1.1) have a Laplacian which is a function of locally bounded mean oscillation, and thus prove Theorem 1.1.

Proof of Theorem 1.1. We fix $r > 0$ and $x_0 \in \Omega$ such that the ball $B_r(x_0) \Subset \Omega$, and we consider the function h that solves

$$\begin{cases} \Delta^2 h = 0 & \text{in } B_r(x_0), \\ h = u & \text{on } \partial B_r(x_0), \\ \nabla h = \nabla u & \text{on } \partial B_r(x_0). \end{cases}$$

The existence of h follows from the Green's formula for biharmonic functions, see page 48 in [GGS10], or by minimizing energy with

$$h - u \in W_0^{2,2}(B_r(x_0)). \tag{3.1}$$

We also extend h outside $B_r(x_0)$ to be equal to u in $\Omega \setminus B_r(x_0)$. We observe that the function h is an admissible competitor for u, since

$$h \in W^{2,2}(\Omega). \tag{3.2}$$

Indeed, if $v := h - u$, we see from (3.1) and the extension results in classical Sobolev spaces (see e.g. Proposition IX.18 in [Bre83]) that $v \in W^{2,2}(\Omega)$. Since $u \in W^{2,2}(\Omega)$, the claim in (3.2) plainly follows.

Then, by the minimality of u, we have that $J[u] \leqslant J[h]$, that is

$$\int_{B_r(x_0)} |\Delta u|^2 + \chi_{\{u>0\}} \leqslant \int_{B_r(x_0)} |\Delta h|^2 + \chi_{\{h>0\}},$$

which in turn yields

$$\int_{B_r(x_0)} |\Delta u|^2 - |\Delta h|^2 \leqslant Cr^n, \tag{3.3}$$

for some $C > 0$. Also, by (3.1), and since $\Delta^2 h = 0$ in $B_r(x_0)$, we get

$$\begin{aligned}
\int_{B_r(x_0)} |\Delta u|^2 - |\Delta h|^2 &= \int_{B_r(x_0)} (\Delta u - \Delta h)(\Delta u + \Delta h) \\
&= \int_{B_r(x_0)} (\Delta u - \Delta h)\Delta u \\
&= \int_{B_r(x_0)} |\Delta u - \Delta h|^2.
\end{aligned}$$

From this and (3.3), we obtain that

$$\int_{B_r(x_0)} |\Delta u - \Delta h|^2 \leqslant Cr^n. \tag{3.4}$$

Now we introduce the notation

$$(\Delta h)_{x_0,r} := \fint_{B_r(x_0)} \Delta u(x)\, dx,$$

and we observe that, by Hölder's inequality,

$$|(\Delta u)_{x_0,r} - (\Delta h)_{x_0,r}|^2 \leqslant \left(\fint_{B_r(x_0)} |\Delta u - \Delta h| \right)^2 \leqslant \fint_{B_r(x_0)} |\Delta u - \Delta h|^2$$

which implies that

$$\int_{B_r(x_0)} |(\Delta h)_{x_0,r} - (\Delta u)_{x_0,r}|^2 \leqslant \int_{B_r(x_0)} |\Delta u - \Delta h|^2. \tag{3.5}$$

Moreover, since the function $H := \Delta h$ is harmonic in $B_r(x_0)$, we have the following Campanato type estimate: there exists $\alpha > 0$ such that, for any $R \geqslant r > 0$ with $B_{2R}(x_0) \Subset \Omega$, there exists a universal constant $C > 0$ such that

$$\fint_{B_r(x_0)} |\Delta h - (\Delta h)_{x_0,r}|^2 \leqslant C \left(\frac{\rho}{R} \right)^\alpha \fint_{B_R} |\Delta h - (\Delta h)_{x_0,R}|^2,$$

see e.g. Theorem 5.1 in [DM93]. Hence, using also the triangle inequality and recalling (3.4) and (3.5),

$$\begin{aligned}
&\int_{B_r(x_0)} |\Delta u - (\Delta u)_{x_0,r}|^2 \\
= \;& \int_{B_r(x_0)} |\Delta u - \Delta h + \Delta h - (\Delta h)_{x_0,r} + (\Delta h)_{x_0,r} - (\Delta u)_{x_0,r}|^2 \\
\leqslant \;& C \left(\int_{B_r(x_0)} |\Delta u - \Delta h|^2 + \int_{B_r(x_0)} |\Delta h - (\Delta h)_{x_0,r}|^2 + \int_{B_r(x_0)} |(\Delta h)_{x_0,r} - (\Delta u)_{x_0,r}|^2 \right) \\
\leqslant \;& C \left(r^n + \left(\frac{r}{R} \right)^\alpha \int_{B_R(x_0)} |\Delta h - (\Delta h)_{x_0,R}|^2 \right)
\end{aligned}$$

$$
\begin{aligned}
&= C\left(r^n + \left(\frac{r}{R}\right)^\alpha \int_{B_R(x_0)} |\Delta h - \Delta u + \Delta u - (\Delta u)_{x_0,R} + (\Delta u)_{x_0,R} - (\Delta h)_{x_0,R}|^2\right)\\
&\leqslant C\left[r^n + \left(\frac{r}{R}\right)^\alpha \left(\int_{B_R(x_0)} |\Delta h - \Delta u|^2 + \int_{B_R(x_0)} |\Delta u - (\Delta u)_{x_0,R}|^2 + \int_{B_R(x_0)} |(\Delta u)_{x_0,R} - (\Delta h)_{x_0,R}|^2\right)\right]\\
&\leqslant C\left[r^n + \left(\frac{r}{R}\right)^\alpha \left(\int_{B_R(x_0)} |\Delta h - \Delta u|^2 + \int_{B_R(x_0)} |\Delta u - (\Delta u)_{x_0,R}|^2\right)\right].
\end{aligned}
$$

Iterating this inequality as in Lemma 3.1 in [DK18] (see also Theorem 1.1 in [DKV17]), we get that

$$
\int_{B_R(x_0)} |\Delta u - (\Delta u)_{x_0,R}|^2 \leqslant CR^n,
$$

for a suitable $C > 0$, which gives the desired result and finishes the proof of Theorem 1.1. □

4. First variation of J, free boundary condition, and proof of Theorem 1.3

In this section, we consider the first variation of the functional in (1.1). Of course, the main problem is to take into account variations performed by a test function whose support intersects the free boundary of u, since in this case the lack of regularity of the characteristic function plays an important role. Therefore, it is useful to know that the set $\{u > 0\}$ is an open subset of Ω, which, in the case of minimizers, follows from the fact that

$$
u \in C^{1,\alpha}_{\text{loc}}(\Omega) \text{ for any } \alpha \in (0,1), \tag{4.1}
$$

which, in turn, follows from the fact that

$$
u \in W^{2,p}_{\text{loc}}(\Omega) \text{ for any } p \in (1,+\infty), \tag{4.2}
$$

in virtue of Theorem 1.1 and the Calderón-Zygmund regularity theory.

The main structural properties of the minimizers which are based on the first variation of the functional are given by the following result:

Lemma 4.1. *Let u be a minimizer of J. Then u is super-biharmonic in Ω and biharmonic in $\{u > 0\} \cup \{u \leqslant 0\}^\circ$, where E° denotes the interior of E.*

Similarly, if u is a one-phase minimizer of J and B is an open ball contained in $\{u \geqslant a\}$, with $a > 0$, then u is biharmonic in B.

Proof. We prove the claims assuming that u is a minimizer (the one-phase problem can be treated similarly). Define $u_\varepsilon := u - \varepsilon\phi$, where $0 \leqslant \phi \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ and ε is a small parameter to be fixed below. Using the comparison of the energies of u and u_ε, and recalling (1.1), we get

$$
\int_\Omega |\Delta u|^2 - |\Delta u - \varepsilon\Delta\phi|^2 \leqslant \int_\Omega \chi_{\{u-\varepsilon\phi>0\}} - \chi_{\{u>0\}}.
$$

Note that $\{u - \varepsilon\phi > 0\} \subset \{u > 0\}$, provided that $\varepsilon > 0$. Consequently, we have that

$$
0 \geqslant \int_\Omega |\Delta u|^2 - |\Delta u - \varepsilon\phi|^2 = 2\varepsilon \int_\Omega \Delta u\, \Delta\phi - \varepsilon^2 \int_\Omega (\Delta u)^2. \tag{4.3}
$$

Dividing both sides of the last inequality by $\varepsilon > 0$ and then letting $\varepsilon \to 0$, we get that

$$
\int_\Omega \Delta u\, \Delta\phi \leqslant 0.
$$

If we take $\phi \in C^\infty_0(\Omega)$, this gives that u is super-biharmonic. In addition, if we suppose that supp$\phi \subset \{u > 0\}$, then from (4.3) we deduce, without any sign assumption on ε, that

$$
\int_\Omega \Delta u\, \Delta\phi = 0,
$$

which completes the proof of Lemma 4.1. □

Concerning the statement of Lemma 4.1, it is interesting to remark that one-phase minimizers are not necessarily super-biharmonic (an explicit counterexample to this fact is discussed on page 17).

The basic analytic structure of the minimizers is then completed by the following result:

Corollary 4.2. *Let u be a minimizer of J. For every bounded subdomain $\Omega' \subset\subset \Omega$, there exists a constant $\widehat{C} > 0$, depending only on* $\mathrm{dist}(\Omega', \partial\Omega)$*, n and λ, such that*

$$\Delta u \geqslant -\widehat{C} \qquad \text{in } \Omega'.$$

Proof. Let $r := \frac{1}{2}\mathrm{dist}(\Omega', \partial\Omega)$ and define the function

$$\phi(y) := \fint_{B_r(y)} \Delta u(x)\, dx.$$

Thanks to (4.2), we see that ϕ is continuous on the compact set $\overline{\Omega'}$. Therefore, there exists $y_0 \in \overline{\Omega'}$ such that $\min_{\overline{\Omega'}} \phi(y) = \phi(y_0)$. Then, for any $y \in \Omega'$,

$$\phi(y) \geqslant \phi(y_0) \geqslant -\fint_{B_r(y_0)} |\Delta u(x)|\, dx =: -\widehat{C}.$$

As a consequence, since u is super-biharmonic, thanks to Lemma 4.1, we obtain the desired estimate. $\square$

Next we compute the first domain variation (for this, we use the notation in which subscripts denote differentiation and superscript denote coordinates).

Lemma 4.3. *Let u be either a minimizer or a one-phase minimizer of J. For any $\phi = (\phi^1, \dots, \phi^n) \in C_0^\infty(\Omega)$ it holds that*

$$2\int_\Omega \Delta u(x) \sum_{m=1}^n \Big(2\nabla u_m(x)\cdot\nabla\phi^m(x) + u_m(x)\Delta\phi^m(x)\Big)\, dx = \int_\Omega \Big(|\Delta u(x)|^2 + \chi_{\{u>0\}}(x)\Big)\mathrm{div}\phi(x)\, dx. \tag{4.4}$$

Proof. Fix $\varepsilon \in \mathbb{R}$ (to be taken with $|\varepsilon|$ small in the sequel). Let

$$u_\varepsilon(x) := u(x + \varepsilon\phi(x)). \tag{4.5}$$

Notice that u_ε is an admissible competitor for u (in case we are dealing with the one-phase problem, observe that $u_\varepsilon \geqslant 0$ if $u \geqslant 0$).

For any $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
\partial_i u_\varepsilon &= \sum_{m=1}^n u_m(\delta_{mi} + \varepsilon\phi_i^m) \\
\text{and } \partial_{ii} u_\varepsilon &= \sum_{m,l=1}^n u_{ml}(\delta_{li} + \varepsilon\phi_i^l)(\delta_{mi} + \varepsilon\phi_i^m) + \sum_{m=1}^n u_m\varepsilon\phi_{ii}^m \\
&= u_{ii} + \varepsilon\left[\sum_{m,l=1}^n \Big(u_{ml}\phi_i^l\delta_{mi} + u_{ml}\phi_i^m\delta_{li}\Big) + \sum_{m=1}^n u_m\phi_{ii}^m\right] + \varepsilon^2\sum_{m,l=1}^n u_{ml}\phi_i^l\phi_i^m \\
&= u_{ii} + \varepsilon\sum_{m=1}^n \Big(2u_{mi}\phi_i^m + u_m\phi_{ii}^m\Big) + \varepsilon^2\sum_{m,l=1}^n u_{ml}\phi_i^l\phi_i^m.
\end{aligned}$$

After the change of variable $y := x + \varepsilon\phi(x)$, we get

$$\begin{aligned}
&J[u_\varepsilon] \\
=\ &\int_\Omega \Bigg\{\Bigg|\sum_{i=1}^n \Big[u_{ii}(x+\varepsilon\phi(x)) + \varepsilon\sum_{m=1}^n \Big(2u_{mi}(x+\varepsilon\phi(x))\phi_i^m(x) + u_m(x+\varepsilon\phi(x))\phi_{ii}^m(x)\Big)\Big] + o(\varepsilon)\Bigg|^2 \\
&\qquad + \chi_{\{u>0\}}(x+\varepsilon\phi(x))\Bigg\}\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \left\{ \left| \sum_{i=1}^n \left[u_{ii}(y) + \varepsilon \sum_{m=1}^n \Big(2u_{mi}(y)\phi_i^m(y) + u_m(y)\phi_{ii}^m(y) \Big) \right] + o(\varepsilon) \right|^2 + \chi_{\{u>0\}}(y) \right\} \Big(1 - \varepsilon \mathrm{div}\phi(y) + o(\varepsilon) \Big)\, dy \\
&= \int_\Omega \left\{ \left| \sum_{i=1}^n u_{ii}(y) + \varepsilon \sum_{i,m=1}^n \Big(2u_{mi}(y)\phi_i^m(y) + u_m(y)\phi_{ii}^m(y) \Big) \right|^2 + \chi_{\{u>0\}}(y) \right\} \Big(1 - \varepsilon \mathrm{div}\phi(y) \Big)\, dy + o(\varepsilon) \\
&= \int_\Omega \left\{ \sum_{i,j=1}^n u_{ii}(y)u_{jj}(y) + 2\varepsilon \sum_{i,j,m=1}^n \Big(2u_{jj}(y)u_{mi}(y)\phi_i^m(y) + u_{jj}(y)u_m(y)\phi_{ii}^m(y) \Big) + \chi_{\{u>0\}}(y) \right\} \\
&\qquad \cdot \Big(1 - \varepsilon \mathrm{div}\phi(y) \Big)\, dy + o(\varepsilon) \\
&= J[u] - \varepsilon \int_\Omega \left\{ \Big(|\Delta u(y)|^2 + \chi_{\{u>0\}}(y) \Big) \mathrm{div}\phi(y) - 2\Delta u(y) \sum_{m=1}^n \Big(2\nabla u_m(y) \cdot \nabla \phi^m(y) + u_m(y)\Delta\phi^m(y) \Big) \right\} dy + o(\varepsilon).
\end{aligned}
$$

Thus taking the derivative in ε and evaluating it at $\varepsilon = 0$ we obtain (4.4), as desired. □

As a consequence of Lemma 4.3, we obtain the free boundary condition of Theorem 1.3:

Proof of Theorem 1.3. We use the notation

$$
\begin{aligned}
& g(x) := |\Delta u(x)|^2 + \chi_{\{u>0\}}(x), \\
& G^m(x) := \Delta u(x) \nabla u_m(x) \\
\text{and} \qquad & H^m(x) := \Delta u(x) u_m(x)
\end{aligned}
$$

for each $m \in \{1, \dots, n\}$. By (4.4) and (1.7), we know that

$$
\begin{aligned}
0 &= \int_\Omega \left(g\, \mathrm{div}\phi - 4 \sum_{m=1}^n G_m \cdot \nabla\phi^m - 2 \sum_{m=1}^n H^m \Delta\phi^m \right) \\
&= \lim_{\varepsilon \to 0} \int_{\Omega \cap \{|u|>\varepsilon\}} \left(g\, \mathrm{div}\phi - 4 \sum_{m=1}^n G_m \cdot \nabla\phi^m - 2 \sum_{m=1}^n H^m \Delta\phi^m \right) \\
&= \lim_{\varepsilon \to 0} \int_{\Omega \cap \{|u|>\varepsilon\}} \left(\mathrm{div}(g\, \phi) - 4 \sum_{m=1}^n \mathrm{div}(\phi^m G_m) - 2 \sum_{m=1}^n \mathrm{div}(H^m \nabla\phi^m) \right. \\
&\qquad \left. + 4 \sum_{m=1}^n \phi^m \mathrm{div} G_m + 2 \sum_{m=1}^n \nabla H^m \cdot \nabla\phi^m - \nabla g \cdot \phi \right).
\end{aligned}
\tag{4.6}
$$

We remark that, in $\{|u| > \varepsilon\}$,

$$
\begin{aligned}
& 4 \sum_{m=1}^n \phi^m \mathrm{div} G_m + 2 \sum_{m=1}^n \nabla H^m \cdot \nabla\phi^m - \nabla g \cdot \phi \\
&= \sum_{m=1}^n \Big(4\phi^m \big(\nabla\Delta u \cdot \nabla u_m + \Delta u \Delta u_m \big) + 2\big(u_m \nabla\Delta u + \Delta u \nabla u_m \big) \cdot \nabla\phi^m - 2\Delta u \Delta u_m \phi^m \Big) \\
&= \sum_{m=1}^n \Big(4\nabla\Delta u \cdot \nabla u_m \phi^m + 2\Delta u \Delta u_m \phi^m + 2\big(u_m \nabla\Delta u + \Delta u \nabla u_m \big) \cdot \nabla\phi^m \Big) \\
&= \sum_{m=1}^n \Big(4\nabla\Delta u \cdot \nabla u_m \phi^m + 2\Delta u \Delta u_m \phi^m + 2\mathrm{div}\Big(\phi^m \big(u_m \nabla\Delta u + \Delta u \nabla u_m \big) \Big) - 2\mathrm{div}\big(u_m \nabla\Delta u + \Delta u \nabla u_m \big) \phi^m \Big) \\
&= 2 \sum_{m=1}^n \Big(\mathrm{div}\Big(\phi^m \big(u_m \nabla\Delta u + \Delta u \nabla u_m \big) \Big) - u_m \Delta^2 u \phi^m \Big) \\
&= 2 \sum_{m=1}^n \mathrm{div}\Big(\phi^m \big(u_m \nabla\Delta u + \Delta u \nabla u_m \big) \Big),
\end{aligned}
$$

by virtue of Lemma 4.1. As a consequence, we see that

$$\begin{aligned}&\int_{\Omega\cap\{|u|>\varepsilon\}}\left(4\sum_{m=1}^{n}\phi^m\mathrm{div}G_m+2\sum_{m=1}^{n}\nabla H^m\cdot\nabla\phi^m-\nabla g\cdot\phi\right)\\ =&\quad 2\sum_{m=1}^{n}\int_{\Omega\cap\{|u|>\varepsilon\}}\mathrm{div}\Big(\phi^m\big(u_m\nabla\Delta u+\Delta u\nabla u_m\big)\Big)\\ =&\quad 2\sum_{m=1}^{n}\int_{\partial(\Omega\cap\{|u|>\varepsilon\})}\phi^m\big(u_m\nabla\Delta u+\Delta u\nabla u_m\big)\cdot\nu,\end{aligned}$$

where ν is the exterior normal to $\Omega\cap\{|u|>\varepsilon\}$. Hence, using this information in (4.6), we obtain that

$$\begin{aligned}0&=\lim_{\varepsilon\to0}\int_{\partial(\Omega\cap\{|u|>\varepsilon\})}\left(g\,\phi\cdot\nu-\sum_{m=1}^{n}\Big(4\phi^mG_m\cdot\nu+2H^m\nabla\phi^m\cdot\nu-2\phi^m\big(u_m\nabla\Delta u+\Delta u\nabla u_m\big)\cdot\nu\Big)\right)\\ &=\lim_{\varepsilon\to0}\int_{\partial(\Omega\cap\{|u|>\varepsilon\})}\left(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)\phi\cdot\nu-2\sum_{m=1}^{n}\Big(\phi^m\big(\Delta u\nabla u_m-u_m\nabla\Delta u\big)\cdot\nu+\Delta u u_m\nabla\phi^m\cdot\nu\Big)\right).\end{aligned}$$

This gives (1.8). Then, to obtain (1.9), one uses the two different scales of the test function ϕ^m and of its derivative. □

Remark 4.4. We point out that if $n=1$, when the free boundary divides regions of positivity and nonpositivity of u, formula (1.9) gives the free boundary conditions

$$\ddot u^+\dot u^+=\ddot u^-\dot u^- \tag{4.7}$$

$$\text{and}\qquad 2\dot u^+\dddot u^+-|\ddot u^+|^2+1=2\dot u^-\dddot u^--|\ddot u^-|^2, \tag{4.8}$$

where

$$u^+:=\max\{u,0\}\quad\text{and}\quad u^-:=\max\{-u,0\}. \tag{4.9}$$

Also, since $u\in W^{2,2}(\Omega)$ and $n=1$, by standard embedding results we already know that $u\in C^1(\Omega)$. This, in view of (4.7), implies that either $\dot u=0$ at a free boundary point, or $\ddot u^+=\ddot u^-$. That is, either u has horizontal tangent at a free boundary point, or it is C^2 across the free boundary point. Hence, from (4.8), we have the following one-dimensional dichotomy for the free boundary points:

$$\text{either: }\dot u=0\text{ and }|\ddot u^+|^2-|\ddot u^-|^2=1, \tag{4.10}$$

$$\text{or: }\dot u\neq0,\ u\text{ is }C^2\text{ across and }\dddot u^+=-\dddot u^--\frac{1}{2\dot u}. \tag{4.11}$$

5. Two examples in dimension 1

Example 1. To better understand Remark 4.4, we can sketch some one-dimensional computations. Namely, we let $n=1$, consider an interval $\Omega:=(0,A)$, with $A>0$, and prescribe the Navier conditions $u(0)=\ddot u(0)=0$, $u(A)=1$ and $\ddot u(A)=0$. We look for one-phase minimizers of J with such boundary conditions.

In this case, by the finiteness of the energy and Sobolev embedding, we know that the one-phase minimizer is $C^1(0,A)$; also the free boundary points are minimal point for u, and therefore

$$\dot u=0\text{ at any free boundary point.} \tag{5.1}$$

Accordingly, condition (4.10) prescribes that

$$\ddot u^+=1. \tag{5.2}$$

Let us see how such condition emerges from energy considerations. We suppose that the problem develops a free boundary and we denote by $a\in(0,A)$ the largest free boundary point, i.e. $u(a)=0$ and $u>0$ in (a,A). From Lemma 4.1, we know that $\ddddot u=0$ in (a,A) and so u is a polynomial of degree 3 in (a,A). Consequently, we can write, for any $x\in(a,A)$,

$$u(x)=\alpha(x-a)+\beta(x-a)^2+\gamma(x-a)^3.$$

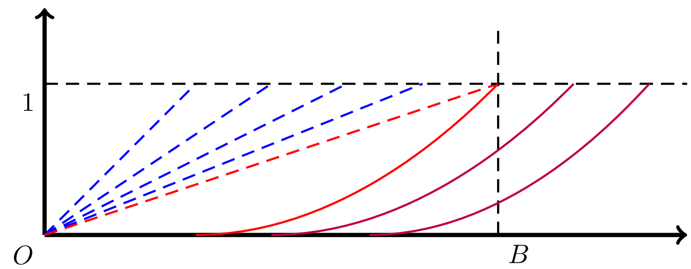

FIGURE 1. *The minimizers of a one-dimensional one-phase problem, in dependence of the right endpoint.*

Then, recalling (5.1), we conclude that $\alpha = 0$. Imposing the boundary conditions at the point $x = A$, we find that

$$\beta = \frac{3}{2(A-a)^2} \qquad \text{and} \qquad \gamma = -\frac{1}{2(A-a)^3},$$

and therefore

$$u(x) = \frac{3(x-a)^2}{2(A-a)^2} - \frac{(x-a)^3}{2(A-a)^3}. \tag{5.3}$$

The goal is then to choose $a \in (0, A)$ in order to minimize the energy contribution of u in (a, A), namely we want to minimize the function

$$\begin{aligned} \Phi(a) \quad &:= \quad \int_a^A |\ddot{u}(x)|^2\,dx + (A-a) \\ &= \quad \int_a^A \left| \frac{3}{(A-a)^2} - \frac{3(x-a)}{(A-a)^3} \right|^2 dx + (A-a) \\ &= \quad 9\int_a^A \left| \frac{(A-a)-(x-a)}{(A-a)^3} \right|^2 dx + (A-a) \\ &= \quad \frac{9}{(A-a)^6} \int_a^A |A-x|^2\,dx + (A-a) \\ &= \quad \frac{3}{(A-a)^3} + (A-a), \end{aligned}$$

which attains its minimum for

$$a = A - \sqrt{3}. \tag{5.4}$$

That is, comparing with the linear function $\ell(x) := \frac{x}{A}$, we have that

$$A = J[\ell] \geqslant J[u] \geqslant \Phi(a) \geqslant \Phi(A-\sqrt{3}) = \frac{1}{\sqrt{3}} + \sqrt{3}.$$

This means that when $A < \frac{1}{\sqrt{3}} + \sqrt{3} =: B$, the problem does not develop any free boundary; when $A = B$ the problem has two minimizers, and when $A > B$ the minimizer in (5.3) becomes

$$u(x) = \frac{(x-a)^2}{2} - \frac{(x-a)^3}{2 \cdot 3^{3/2}}, \tag{5.5}$$

for which $\ddot{u}(a^+) = 1$. This checks (5.2) in this case.

The description of the different one-phase minimizers in dependence of the endpoint A is sketched in Figure 1.

It is also worth pointing out that

(5.6) the one-phase minimizers described here are *not* super-biharmonic, and this creates a major difference with respect to the case of minimizers, compare with Lemma 4.1: indeed, if $\varphi \in C_0^\infty([0,A],[0,+\infty))$ and $A > \frac{1}{\sqrt{3}} + \sqrt{3}$, from (5.4) and (5.5) we see that

$$\begin{aligned}\int_0^A \ddot{u}\ddot{\varphi} &= \int_a^A \left(1 - \frac{x-a}{\sqrt{3}}\right)\ddot{\varphi} = \left(1 - \frac{A-a}{\sqrt{3}}\right)\dot{\varphi}(A) - \dot{\varphi}(a) - \int_a^A \frac{d}{dx}\left(1 - \frac{x-a}{\sqrt{3}}\right)\dot{\varphi}\\ &= 0 - \dot{\varphi}(a) + \frac{1}{\sqrt{3}}\int_a^A \dot{\varphi} = -\dot{\varphi}(a) - \frac{\varphi(a)}{\sqrt{3}},\end{aligned}$$

which has no sign, thus proving (5.6).

Example 2. Having clarified condition (4.10) in a concrete example, we aim now at clarifying the role of condition (4.11). Such condition is, in a sense, more unusual, since it prescribes the matching of the second derivatives at the free boundary points with nontrivial slopes, with the bulk term of the energy producing a discontinuity on the third derivatives.

To understand this phenomenon in a concrete example, we fix a small parameter $\varepsilon > 0$ and minimize the energy functional

$$J[u] = \int_{-1}^1 \left(|\ddot{u}(x)|^2 + \varepsilon\chi_{\{u>0\}}(x)\right)\,dx,$$

subject to the Navier conditions

$$u(-1) = -1, \qquad \ddot{u}(-1) = 0, \qquad u(1) = 1, \qquad \ddot{u}(1) = 0. \tag{5.7}$$

If we call u_ε such minimizer, we can bound the energy of u_ε with that of the identity function. This produces a uniform bound for u_ε in $W^{2,2}((-1,1))$, which implies that u_ε converges in $C^1((-1,1))$ to the identity function as $\varepsilon \to 0$. Consequently, for a fixed and small $\varepsilon > 0$, we can find some $a \in (-1,1)$, which depends on ε, such that

$$u_\varepsilon(x) = \begin{cases} \underline{\alpha}(a-x) + \underline{\beta}(a-x)^2 + \underline{\gamma}(a-x)^3 & \text{if } x \in (-1,a),\\[2ex] \overline{\alpha}(x-a) + \overline{\beta}(x-a)^2 + \overline{\gamma}(x-a)^3 & \text{if } x \in [a,1).\end{cases}$$

The condition that $u_\varepsilon \in C^1((-1,1))$ (with derivative close to 1 when ε is small) implies that $-\underline{\alpha} = \overline{\alpha} = \alpha$, for some $\alpha > 0$ (which depends on ε and it is close to 1 when ε is small). Imposing the boundary conditions in (5.7), we find

$$\underline{\beta} = -\frac{3(1-\alpha(1+a))}{2(1+a)^2}, \qquad \underline{\gamma} = \frac{1-\alpha(1+a)}{2(1+a)^3}, \qquad \overline{\beta} = \frac{3(1-\alpha(1-a))}{2(1-a)^2}, \qquad \overline{\gamma} = \frac{\alpha(1-a)-1}{2(1-a)^3}. \tag{5.8}$$

Therefore, the energy of u_ε corresponds to the function

$$\begin{aligned}\Psi(a,\alpha) &:= J[u_\varepsilon]\\ &= \int_{-1}^a |2\underline{\beta} + 6\underline{\gamma}(a-x)|^2\,dx + \int_a^1 |2\overline{\beta} + 6\overline{\gamma}(x-a)|^2\,dx + \varepsilon\,(1-a)\\ &= \left(\frac{3(1-\alpha(1+a))}{(1+a)^3}\right)^2 \int_{-1}^a |1+x|^2\,dx + \left(\frac{3(1-\alpha(1-a))}{(1-a)^3}\right)^2 \int_a^1 |1-x|^2\,dx + \varepsilon\,(1-a)\\ &= \frac{3(1-\alpha(1+a))^2}{(1+a)^3} + \frac{3(1-\alpha(1-a))^2}{(1-a)^3} + \varepsilon\,(1-a).\end{aligned}$$

Thus, we have to minimize such function for $(a,\alpha) \in (-1,1) \times (0,+\infty)$, and in fact we know that such minimum is localized at $(0,1)$ when $\varepsilon = 0$. Therefore, to find the minima of Ψ, we solve the system

$$\begin{cases} 0 = \partial_a\Psi = \dfrac{12a\big(\alpha a^4(\alpha+2) + 2a^2(2\alpha - \alpha^2 + 3) + \alpha^2 - 6\alpha + 6\big)}{(1-a^2)^4} - \varepsilon,\\[2ex] 0 = \partial_\alpha\Psi = 12\,\dfrac{\alpha - 1 - a^2(1+\alpha)}{(1-a^2)^2}.\end{cases} \tag{5.9}$$

The latter equation produces

$$a^2 = \frac{\alpha-1}{1+\alpha}. \tag{5.10}$$

We notice that, by (5.8),

$$\frac{2}{3}\left(\overline{\beta}-\underline{\beta}\right)=\frac{1-\alpha(1-a)}{(1-a)^2}+\frac{1-\alpha(1+a)}{(1+a)^2}=\frac{2\big((\alpha+1)a^2-\alpha+1\big)}{(1-a^2)^2}.$$

Hence, in view of (5.10),

$$\frac{2}{3}\left(\overline{\beta}-\underline{\beta}\right)=\frac{2\left((\alpha+1)\frac{\alpha-1}{1+\alpha}-\alpha+1\right)}{(1-a^2)^2}=\frac{2\,(\alpha-1-\alpha+1)}{(1-a^2)^2}=0,$$

and so $\overline{\beta}=\underline{\beta}$. This says that the second derivatives match at the free boundary point, in agreement with the condition in (4.11).

In addition, by (5.8),

$$\begin{aligned}4\alpha(\overline{\gamma}+\underline{\gamma})&=2\alpha\left(\frac{\alpha(1-a)-1}{(1-a)^3}+\frac{1-\alpha(1+a)}{(1+a)^3}\right)\\&=-\frac{4\alpha a\,(a^2(2\alpha+1)-2\alpha+3)}{(1-a^2)^3}=-\frac{4\alpha a\,(-2\alpha a^4-a^4+4\alpha a^2-2a^2-2\alpha+3)}{(1-a^2)^4}.\end{aligned}\tag{5.11}$$

On the other hand, the first equation in (5.9) says that

$$\frac{12a}{(1-a^2)^4}=\frac{\varepsilon}{\alpha a^4(\alpha+2)+2a^2(2\alpha-\alpha^2+3)+\alpha^2-6\alpha+6}.$$

Using this information in (5.11), we deduce that

$$12\alpha(\overline{\gamma}+\underline{\gamma})=-\frac{\varepsilon\,\alpha\,(-2\alpha a^4-a^4+4\alpha a^2-2a^2-2\alpha+3)}{\alpha a^4(\alpha+2)+2a^2(2\alpha-\alpha^2+3)+\alpha^2-6\alpha+6}.\tag{5.12}$$

Moreover, in view of (5.10),

$$-2\alpha a^4-a^4+4\alpha a^2-2a^2-2\alpha+3=\frac{4}{(1+\alpha)^2}$$

$$\text{and}\qquad \alpha a^4(\alpha+2)+2a^2(2\alpha-\alpha^2+3)+\alpha^2-6\alpha+6=\frac{4\alpha}{(1+\alpha)^2}.$$

Hence, we insert these identities into (5.12) and we find that

$$2\dot u(a)\left(\dddot u(a^+)-\dddot u(a^-)\right)=12\alpha(\overline{\gamma}+\underline{\gamma})=-\frac{\varepsilon\,\alpha\,\frac{4}{(1+\alpha)^2}}{\frac{4\alpha}{(1+\alpha)^2}}=-\varepsilon,$$

in agreement with the third derivative prescription in (4.11), according to the notation in (4.9).

6. Mechanical interpretation of the free boundary condition (4.11)

In the classical description of the displacement of a thin beam, one assumes that the energy density stored by bending the beam is proportional to the square of the curvature. Namely, supposing that the beam takes the form of a small graphical deformation $u:[0,1]\to\mathbb{R}$ from a horizontal segment, with endpoints normalized at 0 and 1, such energy takes the form of

$$J_1[u]=\frac{\kappa}{2}\int_0^1\frac{|\ddot u(x)|^2}{(1+|\dot u(x)|^2)^3}\,\sqrt{1+|\dot u(x)|^2}\,dx,\tag{6.1}$$

being the first term the square of the curvature and the second the length element. The parameter $\kappa>0$ takes into account the stiffness of the specific material of the beam. Roughly speaking, the rationale of (6.1) is that the rigidity of the material will try to prevent the beam to increase its curvature (with a quadratic law per unit length). For small deformations of a beam, the terms $|\dot u(x)|^2$ are often supposed to be negligible, hence (6.1) is replaced by

$$J_1[u]=\frac{\kappa}{2}\int_0^1|\ddot u(x)|^2\,dx.\tag{6.2}$$

We refer to Section 1.1.1 in [GGS10] and the references therein for additional information on the energy theory of thin beams.

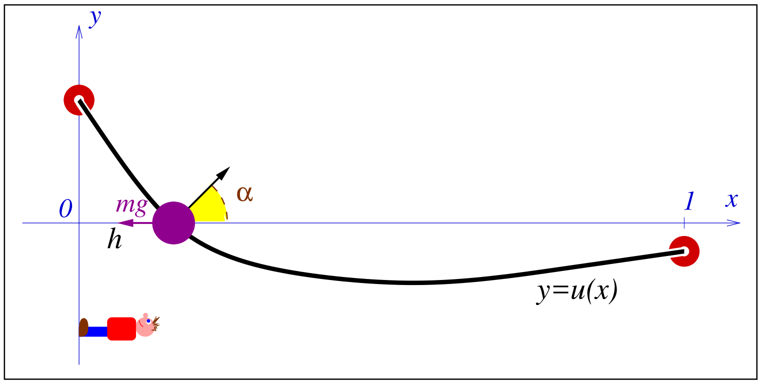

FIGURE 2. *A simple one-dimensional mechanical realization of* (1.1).

We now consider a beam of negligible mass and a particle of mass m in a gravitational field with acceleration g, see[1] Figure 2.

With respect to Figure 2, we notice that the height $h > 0$ of the particle corresponds to the one-dimensional measure of the set $\{u > 0\}$, being the beam represented by the graph $\{y = u(x),\ x \in [0, 1]\}$. Hence, in this setting, the gravitation potential energy of the particle is

$$J_2[u] = mgh = mg \int_0^1 \chi_{\{u>0\}}(x)\, dx.$$

From this and (6.2), we obtain that the full energy of the system is given by

$$J[u] = J_1[u] + J_2[u] = \int_0^1 \frac{\kappa}{2}\, |\ddot{u}(x)|^2 + mg\, \chi_{\{u>0\}}(x)\, dx.$$

Of course, the functional in (1.1) corresponds to the choice

$$\kappa = 2, \qquad m = 1, \qquad g = 1. \tag{6.3}$$

In a balanced configuration, at points $x \neq h$, the beam is free and so it satisfies the equation $\ddddot{u}(x) = 0$. On the other hand, at the point h, the weight of the point mass needs to be balanced by the force produced by the stiffness of the beam, that is (in the distributional sense)

$$\kappa \ddddot{u} + mg\, \Xi = 0, \tag{6.4}$$

where Ξ is the variation of the measure of $\{u > 0\}$ (which is a distribution concentrated at the point h). That is, if $u(h) = 0$ and $\dot{u}(h) \neq 0$, given a test function φ and denoting by $h_\varepsilon = h + \varepsilon\tilde{h} + o(\varepsilon)$ the point such that $(u + \varepsilon\varphi)(h_\varepsilon) = 0$, we have that

$$\begin{aligned} \int_0^1 \Xi(x)\, \varphi(x)\, dx &= \lim_{\varepsilon\to 0} \frac{J_2[u+\varepsilon\varphi] - J_2[u]}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \int_0^1 \frac{\chi_{\{u+\varepsilon\varphi>0\}}(x) - \chi_{\{u>0\}}(x)}{\varepsilon}\, dx = \lim_{\varepsilon\to 0} \frac{h_\varepsilon - h}{\varepsilon} = \tilde{h}. \end{aligned} \tag{6.5}$$

Also, we have that

$$\begin{aligned} 0 &= (u + \varepsilon\varphi)(h_\varepsilon) = u(h + \varepsilon\tilde{h} + o(\varepsilon)) + \varepsilon\varphi(h + \varepsilon\tilde{h} + o(\varepsilon)) \\ &= u(h) + \varepsilon\dot{u}(h)\,\tilde{h} + \varepsilon\varphi(h) + o(\varepsilon) = \varepsilon\Big(\dot{u}(h)\,\tilde{h} + \varphi(h)\Big) + o(\varepsilon), \end{aligned}$$

which gives that $\tilde{h} = -\varphi(h)/\dot{u}(h)$. Therefore, we deduce from (6.5) that

$$\int_0^1 \Xi(x)\, \varphi(x) = -\frac{\varphi(h)}{\dot{u}(h)},$$

[1] Of course for a "real" observer, the x-axis in Figure 2 would be "vertical". We prefer to draw the picture consistently with the mathematical formulation in (1.1) and thus to follow the standard convention of placing the x-axis "horizontally".

and so

$$\Xi = -\frac{\delta_h}{\dot{u}},$$

where δ_h is the Dirac's Delta at the point h. By inserting this into (6.4) we find that

$$\frac{d}{dx}\dddot{u} = \ddddot{u} = -\frac{mg\,\Xi}{\kappa} = \frac{mg\,\delta_h}{\kappa\,\dot{u}}$$

which is compatible with

$$\dddot{u}(h^+) - \dddot{u}(h^-) = \frac{mg}{\kappa\,\dot{u}(h)}. \tag{6.6}$$

In particular, with the choices in (6.3), we obtain the condition

$$\dddot{u}(h^+) - \dddot{u}(h^-) = \frac{1}{2\dot{u}(h)},$$

which is (4.11) (notice indeed that h^+ comes in Figure 2 from the negative part of u and h^- comes in Figure 2 from the positive part of u, therefore, by (4.9), we have that $\dddot{u}(h^{\pm}) = \mp\dddot{u}^{\mp}$).

It is interesting to observe that there is also a derivation based on elementary dynamics of (6.6). Namely, the stiffness of the beam produces a force at the point $(h, u(h))$, normal to the beam for small displacements, whose intensity is minus κ times the second variation of the curvatures, that is

$$-\kappa\frac{d^2}{dx^2}\ddot{u}(h) = -\kappa\ddddot{u}(h).$$

In the setting of Figure 2, the projection of this force along the x-axis is

$$-\kappa\ddddot{u}(h)\,\cos\alpha \simeq -\kappa\ddddot{u}(h)\,\cot\alpha = \kappa\ddddot{u}(h)\,\dot{u}(h),$$

where the small displacement ansatz has been used. At the equilibrium, this must balance the weight of the point mass, therefore we obtain that

$$\kappa\ddddot{u}(h)\,\dot{u}(h) = mg,$$

that is (6.6) at the point $x = h$.

7. A dichotomy argument, and proof of Theorem 1.7

Before proving Theorem 1.7, we show a result concerning the convergence of the blow-up sequence of a minimizer.

Lemma 7.1. *Let $D \Subset \Omega$. Let u_k, with $k \in \mathbb{N}$, be a sequence of minimizers of*

$$\int_D |\Delta u_k|^2 + M_k\chi_{\{u_k>0\}},$$

with $M_k \in (0,1)$, such that $0 \in \partial\{u_k > 0\}$ and $|\nabla u_k(0)| = 0$.

Fix $R > 0$ such that $B_{4R} \Subset D$, and suppose that

$$\sup_{B_{4R}} u_k \leqslant C_0(R), \tag{7.1}$$

for any $k \in \mathbb{N}$, for some $C_0(R) > 0$. Then, there exists a positive constant $C(R)$, independent of k, such that

$$\|u_k\|_{W^{2,2}(B_R)} \leqslant C(R), \tag{7.2}$$

$$\textit{and} \qquad \|\Delta u_k\|_{BMO(B_R)} \leqslant C(R), \tag{7.3}$$

for any $k \in \mathbb{N}$.

Furthermore, there exists a blow-up limit $u_0 : \mathbb{R}^n \to \mathbb{R}$ such that, up to subsequences, as $k \to +\infty$, $u_k \to u_0$ in $W^{2,2}_{\rm loc}(\mathbb{R}^n) \cap C^{1,\alpha}_{\rm loc}(\mathbb{R}^n)$, for any $\alpha \in (0,1)$.

Proof. To check (7.2), we observe that, in virtue of Lemma 4.1,

$$\int_{B_{2R}} \Delta u_k \, \Delta\phi \leqslant 0, \tag{7.4}$$

for any $\phi \in W_0^{2,2}(B_{2R})$. Now, we take $\xi \in C_0^\infty(B_{2R}, [0,1])$ such that

$$\xi = 1 \text{ in } B_R, \qquad |\nabla\xi| \leqslant \frac{C}{R} \qquad \text{and} \qquad |D^2\xi| \leqslant \frac{C}{R^2}, \tag{7.5}$$

for some $C > 0$, we set $m_k := \min_{B_{4R}} u_k$, and we choose $\phi := (u_k - m_k)\xi^2 \geqslant 0$ in (7.4). In this way, setting

$$\begin{aligned} I_1 &:= 2\int_{B_{2R}} \Delta u_k \, \nabla u_k \cdot \nabla\xi^2 \\ \text{and} \qquad I_2 &:= \int_{B_{2R}} (u_k - m_k)\, \Delta u_k \, \Delta\xi^2, \end{aligned}$$

we have that

$$0 \geqslant \int_{B_{2R}} \Delta u_k \, \Delta\big((u_k - m_k)\xi^2\big) = \int_{B_{2R}} (\Delta u_k)^2\xi^2 + I_1 + I_2. \tag{7.6}$$

Now, thanks to Corollary 4.2, we can use the standard method to prove Caccioppoli inequality: namely we take $\eta \in C_0^\infty(B_{4R}, [0,1])$ such that $\eta = 1$ in B_{2R} and $|\nabla\eta| \leqslant \frac{C}{R}$ and we infer from Corollary 4.2 that

$$\begin{aligned} \widehat{C}\int_{B_{4R}} (u_k - m_k)\,\eta^2 &\geqslant -\int_{B_{4R}} \Delta u_k\,(u_k - m_k)\,\eta^2 = \int_{B_{4R}} |\nabla u_k|^2\eta^2 + \int_{B_{4R}} 2\eta\,(u_k - m_k)\,\nabla\eta\cdot\nabla u_k \\ &\geqslant \frac12\int_{B_{4R}} |\nabla u_k|^2\eta^2 - C\int_{B_{4R}} (u_k - m_k)^2|\nabla\eta|^2, \end{aligned}$$

which implies that

$$\int_{B_{2R}} |\nabla u_k|^2 \leqslant \frac{C}{R^2}\int_{B_{4R}} (u_k - m_k)^2 + C\int_{B_{4R}} (u_k - m_k) \tag{7.7}$$

for some $C > 0$, possibly varying from line to line.

Hence, by Young's inequality, (7.5) and (7.7), we get

$$\begin{aligned} |I_1| &\leqslant 2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{1}{\varepsilon}\int_{B_{2R}} |\nabla u_k|^2\,|\nabla\xi|^2\right) \\ &\leqslant 2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon R^2}\int_{B_{2R}} |\nabla u_k|^2\right) \\ &\leqslant 2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon R^4}\int_{B_{4R}} (u_k - m_k)^2 + \frac{C}{R^2}\int_{B_{4R}} (u_k - m_k)\right). \end{aligned} \tag{7.8}$$

Furthermore, noticing that $(u_k - m_k)\Delta u_k|\nabla\xi|^2 \geqslant -\widehat{C}(u_k - m_k)|\nabla\xi|^2$, thanks to Corollary 4.2, and making again use of Young's inequality, we obtain that

$$\begin{aligned} I_2 &= \int_{B_{2R}} (u_k - m_k)\,\Delta u_k\left(2\xi\Delta\xi + |\nabla\xi|^2\right) \\ &\geqslant 2\int_{B_{2R}} (u_k - m_k)\,\Delta u_k\,\xi\,\Delta\xi - \widehat{C}\int_{B_{2R}} (u_k - m_k)|\nabla\xi|^2 \\ &\geqslant -2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{1}{\varepsilon}\int_{B_{2R}} (u_k - m_k)^2(\Delta\xi)^2\right) - \widehat{C}\int_{B_{2R}} (u_k - m_k)|\nabla\xi|^2 \\ &\geqslant -2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon R^4}\int_{B_{2R}} (u_k - m_k)^2\right) - \frac{C}{R^2}\int_{B_{2R}} (u_k - m_k). \end{aligned}$$

From this, (7.6) and (7.8), we conclude that

$$\int_{B_{2R}} (\Delta u_k)^2\xi^2 \quad\leqslant\quad 2\left(\varepsilon\int_{B_{2R}} (\Delta u_k)^2\xi^2 + \frac{C}{\varepsilon R^4}\int_{B_{2R}} (u_k - m_k)^2 + \frac{C}{R^2}\int_{B_{4R}} (u_k - m_k)\right)$$

$$+2\left(\varepsilon\int_{B_{2R}}(\Delta u_k)^2\xi^2+\frac{C}{\varepsilon R^4}\int_{B_{2R}}(u_k-m_k)^2\right)+\frac{C}{R^2}\int_{B_{2R}}(u_k-m_k),$$

which, in turn, implies that

$$(1-4\varepsilon)\int_{B_{2R}}(\Delta u_k)^2\xi^2\leqslant\frac{C}{\varepsilon R^4}\int_{B_{2R}}(u_k-m_k)^2+\frac{C}{R^2}\int_{B_{2R}}(u_k-m_k)\leqslant\frac{C}{\varepsilon}+C,$$

where the last step follows from (7.1). Choosing $\varepsilon=\frac{1}{8}$ and recalling (7.5), we obtain that

$$\int_{B_R}(\Delta u_k)^2\leqslant\int_{B_{2R}}(\Delta u_k)^2\xi^2\leqslant C,$$

up to renaming $C>0$, that does not depend on k. This implies the desired estimate in (7.2).

Moreover, the estimate in (7.3) follows from the BMO estimates in Section 3.

Finally, from the uniform estimate in (7.2), we can apply a customary compactness argument to conclude that there exists a function u_0 such that, up to a subsequence, $u_k\to u_0$ in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n)\cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, for any $\alpha\in(0,1)$, as $k\to+\infty$. This completes the proof of Lemma 7.1. $\square$

With this, we are now in the position of completing the proof of Theorem 1.7.

Proof of Theorem 1.7. We claim that there exist an integer $k_0>0$ and a constant $C>0$, depending only on δ, n and $\mathrm{dist}(D,\Omega)$, such that the following inequality holds:

$$\sup_{B_{2^{-k-1}}(x_0)}|u|\leqslant\max\left\{\frac{C}{2^{2k}},\,\frac{\sup_{B_{2^{-k}}(x_0)}|u|}{2^2},\dots,\frac{\sup_{B_{2^{-k+m}}(x_0)}|u|}{2^{2(m+1)}},\dots,\frac{\sup_{B_1(x_0)}|u|}{2^{2(k+1)}}\right\},\tag{7.9}$$

for any $k\geqslant k_0$.

Indeed, if (7.9) fails, then, for any $j\in\mathbb{N}$, there exist singular free boundary points $x_j\in D$, integers k_j and minimizers u_j such that

$$\sup_{B_{2^{-k_j-1}}(x_j)}|u_j|>\max\left\{\frac{j}{2^{2k_j}},\,\frac{\sup_{B_{2^{-k_j}}(x_j)}|u_j|}{2^2},\dots,\frac{\sup_{B_{2^{-k_j+m}}(x_j)}|u_j|}{2^{2(m+1)}},\dots,\frac{\sup_{B_1(x_j)}|u_j|}{2^{2(k_j+1)}}\right\}.\tag{7.10}$$

We denote by $S_j:=\sup_{B_{2^{-k_j-1}}(x_j)}|u_j|$ and we consider the scaled functions

$$v_j(x):=\frac{u_j(x_j+2^{-k_j}x)}{S_j}.$$

In this way, (7.10) gives that

$$1>\max\left\{\frac{j}{2^{2k_j}\,S_j},\,\frac{\sup_{B_1}|v_j|}{2^2},\dots,\frac{\sup_{B_{2^m}}|v_j|}{2^{2(m+1)}},\dots,\frac{\sup_{B_{2^{k_j}}}|v_j|}{2^{2(k_j+1)}}\right\}.$$

From this, we have that the functions v_j satisfy the following properties:

$$\begin{gathered}\sup_{B_{1/2}}v_j=1,\\[2ex] v_j(0)=|\nabla v_j(0)|=0,\\[2ex] \sup_{B_{2^m}}|v_j|\leqslant 4\cdot 2^{2m},\quad\text{for any } m<k_j,\\[2ex] \sigma_j:=\frac{1}{2^{2k_j}}\frac{1}{S_j}<\frac{1}{j}.\end{gathered}\tag{7.11}$$

We also remark that, from the scaling properties of the functional J, we have that

$$\int_{B_R}|\Delta v_j|^2+\sigma_j^2\chi_{\{v_j>0\}}=2^{k_jn}\sigma_j^2\int_{B_{R2^{-k_j}}(x_j)}|\Delta u_j|^2+\chi_{\{u_j>0\}},$$

for every fixed $R < 2^{k_j}$. This says that v_j is a minimizer in B_R among all the functions in $W^{2,2}(B_R)$ whose Navier boundary conditions agree with those of u_j on ∂B_R.

Now, by assumption, u_j is not δ-rank-2 flat at each level $r = 2^{-k}$, for any $k \geqslant 1$, at x_j. As a consequence, v_j is not δ-rank-2 flat in B_1. So, recalling (1.14) and Definition 1.6, this means that

$$h(1,0) = \inf_{p \in P_2} h_{\min}(1, x_0, p) \geqslant \delta. \tag{7.12}$$

Also, we have that condition (7.1) is guaranteed in this case, in view of (7.11). Therefore, from Lemma 7.1, applied here with $M_j := \sigma_j^2$, we know that, up to a subsequence, still denoted by v_j, there exists a function v_∞ such that

$$v_j \to v_\infty \text{ in } W^{2,2}(B_R) \cap C^{1,\alpha}(B_R), \text{ for any } \alpha \in (0,1), \text{ as } j \to +\infty. \tag{7.13}$$

Moreover, we have that $\Delta v_j \in BMO(B_R)$ uniformly. Consequently $v_\infty \in W^{2,2}(B_R) \cap C^{1,\alpha}(B_R)$, for all $\alpha \in (0,1)$, and $\Delta v_\infty \in BMO(B_R)$. Furthermore,

$$\begin{aligned} &\Delta^2 v_\infty = 0 \ \text{ in } \mathbb{R}^n, \qquad \sup_{B_{1/2}} v_\infty = 1, \\ &|v_\infty(x)| \leqslant 8|x|^2 \ \text{ for any } x \in \mathbb{R}^n \\ &\text{and} \qquad v_\infty(0) = |\nabla v_\infty(0)| = 0. \end{aligned} \tag{7.14}$$

Let now $f := \Delta v_\infty$, then we have that f is harmonic in $\mathbb{R}^n$. Moreover, by Lemma A.1 and the second line in (7.14), we see that, for any $r > 0$,

$$\frac{1}{r^n} \int_{B_r} |D^2 v_\infty|^2 \leqslant \frac{C}{r^{n+4}} \int_{B_r} \big(v_\infty - \min_{B_{4r}} v_\infty\big)^2 + \frac{C}{r^{n+2}} \int_{B_r} \big(v_\infty - \min_{B_{4r}} v_\infty\big) \leqslant C,$$

up to renaming $C > 0$. Thus, from the Liouville Theorem we infer that f must be constant, i.e. $\Delta v_\infty = C_0$, for some $C_0 \in \mathbb{R}$.

Consequently, $v_\infty - \frac{C_0}{2n}|x|^2$ is harmonic in $\mathbb{R}^n$ with quadratic growth. Hence, by using the Liouville Theorem once again, we have that $v_\infty(x) = g(x) + \frac{C_0}{2n}|x|^2$, where g is a second order polynomial. Moreover, since $\nabla v_\infty(0) = 0$, we deduce that $g = cp$, for some $c \in \mathbb{R}$ and $p \in P_2$ (recall (1.11)).

Therefore, we can write

$$v_\infty(x) = x \cdot Ax,$$

for some constant and symmetric matrix A. Consequently, recalling the notation in (1.12),

$$\partial\{v_\infty > 0\} = S(p, 0) \tag{7.15}$$

for some $p \in P_2$. On the other hand, from our construction in (7.12), we have that

$$\mathrm{HD}\big(\partial\{v_j > 0\} \cap B_1, S(p,0) \cap B_1\big) \geqslant \delta$$

(recall the definitions of HD and $h_{\min}$ in (1.10) and (1.13), respectively). As a consequence, there exist points $z_j \in \partial\{v_j > 0\} \cap B_1$ such that

$$\operatorname{dist}(z_j, S(p,0)) \geqslant \delta. \tag{7.16}$$

Now we extract a converging sequence, still denoted z_j, such that $z_j \to z_0$ as $j \to +\infty$, and we see from the uniform convergence of v_j given in (7.13) that $v_\infty(z_0) = 0$, which implies that $z_0 \in S(p,0)$, thanks to (7.15). On the other hand, we also have that $\operatorname{dist}(z_0, S(p,0)) \geqslant \delta$, in virtue of (7.16). Therefore, we reach a contradiction, and so the proof of Theorem 1.7 is finished. ☐

8. Non-degeneracy, and proof of Theorems 1.8 and 1.10

In this section we deal with weak and strong nondegeneracy properties of the minimizers. Due to the lack of Harnack inequalities for biharmonic functions, the strong nondegeneracy result does not follow immediately from the weak one, unless we impose some additional conditions on the set $\{u > 0\}$.

8.1. **Weak nondegeneracy, and proof of Theorem 1.8.** Here we prove the weak nondegeneracy for u^+, according to the statement in Theorem 1.8.

Proof of Theorem 1.8. We observe that the claim in $\mathbf{1}^\circ$ implies the one in $\mathbf{2}^\circ$, therefore we focus on the proof of the first claim.

After rescaling u by defining $r^{-2}u(x_0+rx)$, we may assume without loss of generality that $r=1$ and $x_0=0$. Also, denote by

$$\gamma := \sup_{B_1} u. \tag{8.1}$$

Let $\psi \in C^\infty(\mathbb{R}^n,[0,1])$ such that $\psi = 0$ in $B_{\frac{1}{16}}$, $\psi > 0$ in $\mathbb{R}^n \setminus \overline{B_{\frac{1}{16}}}$ and $\psi = 1$ in $\mathbb{R}^n \setminus B_{\frac{1}{8}}$. Set $v := \psi u$. Then $u - v \in W_0^{2,2}(B_{\frac{1}{8}})$, and so v is a competitor for u in $B_{\frac{1}{8}}$. Therefore, from the minimality of u we have that

$$\int_{B_{\frac{1}{8}}} |\Delta u|^2 + \chi_{\{u>0\}} \leqslant \int_D |\Delta v|^2 + \chi_{\{v>0\}},$$

where $D := B_{\frac{1}{8}} \setminus \overline{B_{\frac{1}{16}}}$. From this, and recalling the definitions of v and ψ, we obtain that

$$\begin{aligned}
\big|\{u>0\}\cap B_{\frac{1}{16}}\big| &\leqslant \int_{B_{\frac{1}{16}}} |\Delta u|^2 + \chi_{\{u>0\}} \\
&\leqslant \int_D |\Delta v|^2 + \chi_{\{v>0\}} - \int_D |\Delta u|^2 + \chi_{\{u>0\}} \\
&= \int_D |\Delta v|^2 - |\Delta u|^2 \\
&\leqslant \int_D |\Delta v|^2.
\end{aligned}$$

Hence, using Lemma A.1, it follows that

$$\begin{aligned}
\big|\{u>0\}\cap B_{\frac{1}{16}}\big| &\leqslant \int_D (u\Delta\psi + 2\nabla u \nabla\psi + \psi\Delta u)^2 \\
&\leqslant 2\|\psi\|_{C^2(B_{\frac{1}{8}})} \int_D u^2 + 4|\nabla u|^2 + |D^2 u|^2 \\
&\leqslant 8C\|\psi\|_{C^2(B_{\frac{1}{8}})}\, |B_{\frac{1}{8}}|\gamma(1+\gamma),
\end{aligned} \tag{8.2}$$

for some $C>0$, where γ is the quantity introduced in (8.1).

On the other hand, from (1.16), we have that

$$\big|\{u>0\}\cap B_{\frac{1}{16}}\big| \geqslant \theta_*\, |B_{\frac{1}{16}}|.$$

Combining this and (8.2), we conclude that

$$\gamma(1+\gamma) \geqslant \theta_* \frac{|B_{\frac{1}{16}}|}{8C\|\psi\|_{C^2(B_{\frac{1}{8}})}|B_{\frac{1}{8}}|},$$

which implies the desired result. □

8.2. **Whitney's covering.** Here we recall the Whitney's decomposition method, to obtain suitable conditions which imply formula (1.16) (in our setting, this condition will be provided by the forthcoming formula (8.3)). Suppose that $E \subset \mathbb{R}^n$ is a nonempty compact set, then $\mathbb{R}^n \setminus E$ can be represented as a union of closed dyadic cubes Q_j^k with mutually disjoint interiors

$$\mathbb{R}^n \setminus E = \bigcup_{k\in\mathbb{Z}} \bigcup_{j=1}^{N_k} Q_j^k$$

such that

$$c_1 \leqslant \frac{\operatorname{dist}(Q_j^k, E)}{\operatorname{diam} Q_j^k} \leqslant c_2$$

for two universal constants c_1, $c_2 > 0$. Here Q_j^k is a cube with side length equal to 2^{-k}.

Let now $E := \{u \leqslant 0\} \cap \overline{Q_1(x_0)}$, where $Q_1(x_0)$ is the unit cube centered at $x_0 \in \partial\{u > 0\}$, and consider the Whitney's decomposition for $\mathbb{R}^n \setminus E$. Let $k_0 \in \mathbb{N}$ be fixed, and suppose that for every $k \geqslant k_0$ there exists $c > 0$ such that, for some Q_j^k, we have

$$\operatorname{dist}\left(x_0, Q_j^k\right) \leqslant c2^{-k}. \tag{8.3}$$

Then u^+ is strongly nondegenerate at x_0. To see this, for every large k let us take a cube Q_j^k such that (8.3) holds. Then

$$\frac{|\{u > 0\} \cap B_{c2^{-k}}(x_0)|}{|B_{c2^{-k}}|} \geqslant \frac{1}{c^n}.$$

Therefore (1.16) holds and the claim follows from Theorem 1.8.

Definition 8.1. *If (8.3) holds, then we say that $\partial\{u > 0\}$ satisfies a weak c-covering condition at $x_0 \in \partial\{u > 0\}$.*

We remark that the standard c-covering condition, that was introduced in [MV87], is stronger than (8.3) and indeed it requires that

$$\operatorname{dist}\left(x_0, \bigcup_{j=1}^{N_k} Q_j^k\right) \leqslant c2^{-k}.$$

Moreover, it is known that the weak c-covering condition of Definition 8.1 is satisfied by the John domains, see [MV87].

In order to recall the definition of John domain, we let $0 < \alpha \leqslant \beta < \infty$. A domain $D \subset \mathbb{R}^n$ is called an (α, β)-John domain, denoted by $D \in \mathcal{J}(\alpha, \beta)$, if there exists $x_0 \in D$ such that every $x \in D$ has a rectifiable path $\gamma : [0, d] \to D$ with arc length as parameter such that $\gamma(0) = x$, $\gamma(d) = x_0$, $d \leqslant \beta$ and

$$\operatorname{dist}(\gamma(t), \partial D) \geqslant \frac{\alpha}{d} t, \quad \text{for all} \quad t \in [0, d].$$

The point x_0 is called a center of D. A domain D is called a John domain if $D \in \mathcal{J}(\alpha, \beta)$ for some α and β. The class of all John domains in $\mathbb{R}^n$ is denoted by $\mathcal{J}$.

For more on such coverings and applications of Whitney's decompositions we refer to [MV87].

Alternative sufficient geometric conditions on $\{u > 0\}$ guaranteeing the strong nondegeneracy of u can be given. Note that in order to pass from weak to strong nondegeneracy at some $z \in \partial\{u > 0\}$ it is enough to have a small ball $B' \subset B_r(z) \cap \{u > 0\}$ and $c > 0$ such that $\operatorname{diam} B' \geqslant cr$ for every small r, since this guarantees (1.16).

Definition 8.2. *We say that $\partial\{u > 0\}$ satisfies a nonuniform interior cone condition if for every $x \in \partial\{u > 0\}$ there exist a positive number $r_x > 0$ and a cone K_x with vertex at x, such that $B_{r_x}(x) \cap K_x \subset \{u > 0\}$.*

We also say that $\partial\{u > 0\}$ satisfies a uniform interior cone condition if there exist a positive number $r > 0$ and a cone K with vertex at 0, such that for every $x \in \partial\{u > 0\}$ we have that $B_r(x) \cap (x + K) \subset \{u > 0\}$.

From our observation above and Theorem 1.8 we immediately obtain the following result:

Corollary 8.3. *Let u be a minimizer for J in Ω, and $x_0 \in \Omega$. Suppose that $\{u > 0\}$ satisfies the interior cone condition at $x_0 \in \partial\{u > 0\}$, then u is nondegenerate at x_0. Moreover, if $\{u > 0\}$ satisfies the uniform interior cone condition and $B_1 \subset \Omega$, then*

$$\sup_{B_r(z)} u^+ \geqslant C_0 r^2,$$

for any $z \in \partial\{u > 0\} \cap B_1$, for some universal constant $C_0 > 0$.

8.3. **The biharmonic measure, and proof of Theorem 1.10.** In this subsection, we describe the main features of the measure induced by the bi-Laplacian of a minimizer. For this, we observe that, since, by Lemma 4.1, Δu is super-harmonic,

$$\text{there exists a nonnegative measure } \mathcal{M}_u \text{ such that } -\Delta^2 u = \mathcal{M}_u. \tag{8.4}$$

Hence, for any $\psi \in C_0^\infty(\Omega)$, we have that

$$\int_\Omega \mathcal{M}_u \psi = \int_\Omega (-\Delta u)\Delta\psi. \tag{8.5}$$

Recalling the notion of flatness introduced in Definition 1.6, we have the following:

Lemma 8.4. *Let u be a minimizer of the functional J defined in (1.1), let $\delta > 0$ and let $x_0 \in \partial\{u > 0\}$ such that $\nabla u(x_0) = 0$ and $\partial\{u > 0\}$ is not δ-rank-2 flat at x_0 at any level $r > 0$ with $B_r(x_0) \Subset \Omega$. Then,*

$$\mathcal{M}_u(B_r(x_0)) \leqslant Cr^{n-2} \tag{8.6}$$

for any $r > 0$ as above, for some $C > 0$.

Proof. Without loss of generality, we take $x_0 = 0$. We consider a function $\psi_0 \in C_0^\infty(B_2, [0, 1])$, with $\psi_0 = 1$ in B_1, and we let $\psi(x) := \psi_0(x/r)$. In this way, $\psi = 1$ in B_r and $|D^2\psi| \leqslant C/r^2$ for some $C > 0$.

We now exploit (8.5) with such ψ. Then, by Corollary A.2, we have that

$$\mathcal{M}_u(B_r) \leqslant \int_\Omega \mathcal{M}_u\psi = \int_\Omega(-\Delta u)\Delta\psi \leqslant \sqrt{\int_{B_{2r}} |\Delta u|^2}\sqrt{\int_{B_{2r}} |\Delta\psi|^2} \leqslant Cr^{\frac{n}{2}} r^{\frac{n-4}{2}},$$

which implies the desired result, up to renaming $C > 0$. $\square$

We remark that a full counterpart of Lemma 8.4 does not hold for the one-phase problem (in particular $\mathcal{M}_u$ as defined in (8.4) and (8.5) does not need to have a sign, see (5.6)). Nevertheless, the following result holds:

Lemma 8.5. *Let u be a one-phase minimizer of J. Assume that $u \in C^{1,1}(\Omega)$ and $\partial\{u > 0\}$ has null Lebesgue measure. Let $\varphi \in C_0^\infty(B_1, [0, 1])$ with*

$$\int_{B_1} \varphi = 1.$$

For any $\delta > 0$, let

$$\varphi_\delta(x) := \frac{1}{\delta^n}\varphi\left(\frac{x}{\delta}\right),$$

and $u_\delta := u * \varphi_\delta$. Then, for any $\Omega' \Subset \Omega$, we have that

$$\lim_{\delta\to 0}\int_{\Omega'} \Delta^2 u_\delta\, u_\delta = 0.$$

Proof. Let

$$\Gamma_\delta := \bigcup_{p\in\partial\{u>0\}} B_\delta(p).$$

We claim that

$$\text{if } x \in \Omega \setminus \Gamma_\delta, \text{ then } \Delta^2 u(x) = 0. \tag{8.7}$$

To prove this, we argue by contradiction and we suppose that there exists $x \in \Omega \setminus \Gamma_\delta$ such that

$$\Delta^2 u(x) \text{ is either not defined or not null.} \tag{8.8}$$

We observe that

$$\text{there exists } \rho,\, a > 0 \text{ such that } u \geqslant a \text{ in } B_\rho(x). \tag{8.9}$$

Because, if not, for any $k \in \mathbb{N}$, there exists x_k such that $|x-x_k|+u(x_k) \leqslant 1/k$, and thus $u(x) = 0$. Since x lies outside Γ_δ, it cannot be a free boundary point, hence u must vanish in a neighborhood of x. Consequently, $\Delta^2 u$ vanishes in a neighborhood of x, and this is in contradiction with (8.8), thus proving (8.9).

Then, from (8.9) and Lemma 4.1, it follows that u is biharmonic in $B_\rho(x)$. Once again, this is in contradiction with (8.8), and thus the proof of (8.7) is complete.

Now, by taking δ sufficiently small, we suppose that the distance from Ω' to $\partial\Omega$ is larger than δ. Thus, from (8.7) we obtain that, if $x \in \Omega' \setminus \Gamma_{2\delta}$ and $y \in B_\delta$, then $x - y \in \Omega' \setminus \Gamma_\delta$, hence $\Delta^2 u(x-y) = 0$.

Consequently, for every $x \in \Omega' \setminus \Gamma_{2\delta}$,

$$\Delta^2 u_\delta(x) = \int_{B_\delta} \Delta^2 u(x-y)\,\varphi_\delta(y)\,dy = 0.$$

This implies that

$$\int_{\Omega'} \Delta^2 u_\delta\, u_\delta = \int_{\Omega'\cap\Gamma_{2\delta}} \Delta^2 u_\delta\, u_\delta. \tag{8.10}$$

We also remark that

$$\begin{aligned} |\Delta^2 u_\delta(x)| &\leqslant \int_{B_\delta} |u(x-y)|\,|\Delta^2 \varphi_\delta(y)|\,dy = \frac{1}{\delta^{n+4}} \int_{B_\delta} |u(x-y)| \left|\Delta^2 \varphi\left(\frac{y}{\delta}\right)\right| dy \\ &= \frac{1}{\delta^4} \int_{B_1} |u(x-\delta y)| \left|\Delta^2 \varphi(y)\right| dy \leqslant \frac{C}{\delta^4} \int_{B_1} u(x-\delta y)\,dy, \end{aligned} \tag{8.11}$$

for some $C > 0$. Now, if $x \in \Gamma_{2\delta}$ and $y \in B_1$, we have that there exists $p \in \partial\{u > 0\} \subseteq \{u = 0\}$ such that $|p - x| \leqslant 2\delta$ and accordingly $|(x - \delta y) - p| \leqslant |x - p| + \delta \leqslant 3\delta$. Then, in this setting, the regularity of u implies that

$$u(x - \delta y) \leqslant 9\|u\|_{C^{1,1}(\Omega)}\delta^2. \tag{8.12}$$

In particular, recalling (8.11), we find that, if $x \in \Gamma_{2\delta}$,

$$|\Delta^2 u_\delta(x)| \leqslant \frac{C}{\delta^2}, \tag{8.13}$$

up to renaming $C > 0$, also depending on $\|u\|_{C^{1,1}(\Omega)}$.

From (8.12) we also deduce that, if $x \in \Gamma_{2\delta}$,

$$|u_\delta(x)| \leqslant \int_{B_1} u(x-\delta y)\,\varphi(y)\,dy \leqslant 9\|u\|_{C^{1,1}(\Omega)}\delta^2.$$

Using this information and (8.13) we conclude that, if $x \in \Gamma_{2\delta}$,

$$|\Delta^2 u_\delta(x)\, u_\delta(x)| \leqslant C,$$

and therefore

$$\left|\int_{\Omega'\cap\Gamma_{2\delta}} \Delta^2 u_\delta\, u_\delta\right| \leqslant C\,|\Omega' \cap \Gamma_{2\delta}|,$$

up to renaming $C > 0$ once again.

This and (8.10) give that

$$\left|\int_{\Omega'} \Delta^2 u_\delta\, u_\delta\right| \leqslant C\,|\Omega' \cap \Gamma_{2\delta}|.$$

Hence, taking the limit as $\delta \to 0$,

$$\lim_{\delta\to 0}\left|\int_{\Omega'} \Delta^2 u_\delta\, u_\delta\right| \leqslant |\Omega' \cap \partial\{u > 0\}|.$$

This gives the desired result. □

Now we prove a counterpart of (8.6) at nondegenerate points of the free boundary of the minimizers. For this, recalling the setting in formula (1.14), we let $\mathcal{N}_\delta$ be the set of free boundary points x with the property that there exists $r_x > 0$ small enough such that $h(r, x) \geqslant \delta r$ for every $r < r_x$. Moreover, in the spirit of Definition 1.4, we also denote by

$$\mathcal{N}_\delta^{\text{sing}} := \{x \in \mathcal{N}_\delta \text{ s.t. } \nabla u(x) = 0\}.$$

Lemma 8.6. *Let u be a minimizer of J. Let $D \subset \Omega$ and suppose that there exists a constant $\bar{c} > 0$ such that*

$$\liminf_{r\to 0} \frac{\sup_{B_r(x)} |u|}{r^2} \geqslant \bar{c} \tag{8.14}$$

for every $x \in \partial\{u > 0\} \cap \overline{D}$. Then there exists a constant $c_0(\delta) > 0$, depending on n, δ, $\bar{c}$ and $\operatorname{dist}(\overline{D}, \partial\Omega)$, such that

$$\liminf_{r\to 0} \frac{\mathcal{M}_u(B_r(x))}{r^{n-2}} \geqslant c_0(\delta), \qquad \textit{for any } x \in \mathcal{N}_\delta^{sing}. \tag{8.15}$$

Proof. We argue by contradiction. If (8.15) fails, then there exists a sequence $x_j \in \mathcal{N}_\delta^{\rm sing}$ such that

$$\liminf_{r\to 0} \frac{\mathcal{M}_u(B_r(x_j))}{r^{n-2}} < \varepsilon_j \tag{8.16}$$

with $\varepsilon_j \to 0$. Since $x_j \in \mathcal{N}_\delta^{\rm sing}$, there exists a sequence $r_j \to 0$ such that

$$h(r_j, x_j) \geqslant \delta r_j. \tag{8.17}$$

Now we define

$$U_j(x) := \frac{u(x_j + r_j x)}{r_j^2}.$$

By construction, recalling (8.14), we have that $\{U_j\}$ is nondegenerate with quadratic growth, i.e. there exists a constant $C > 0$ independent of j such that

$$\frac{1}{C} R^2 \leqslant \sup_{B_R} |U_j| \leqslant C R^2 \qquad \text{for any } R < \frac{1}{r_j}. \tag{8.18}$$

Moreover, by (8.16) and (8.17), we see that

$$h(1,0) \geqslant \delta \quad \text{and} \quad \mathcal{M}_{U_j}(B_R) \leqslant \varepsilon_j R^{n-2} \to 0 \tag{8.19}$$

for every fixed $R > 0$.

As a consequence, using a customary compactness argument, we can extract a converging subsequence, still denoted by U_j, such that $U_j \to U_0$ locally uniformly as $j \to +\infty$. Then (8.19) translates into

$$h(1,0) \geqslant \delta \quad \text{and} \quad \mathcal{M}_{U_0}(B_R) = 0 \tag{8.20}$$

for every fixed $R > 0$. In other words, in view of (8.18), we have that U_0 is an entire nontrivial biharmonic function with quadratic growth.

On the other hand, applying Corollary A.2 we also have that

$$\int_{B_R} |D^2 u|^2 \leqslant C R^n.$$

This, together with the Liouville Theorem, implies that

$$U_0 \text{ is a quadratic polynomial.} \tag{8.21}$$

Accordingly, there exists $\alpha \in \mathbb{R}$ such that $p := \alpha U_0 \in P_2$ (recall the notation in (1.11)). From (8.20), we conclude that

$$\mathrm{HD}(S(p,0) \cap B_1, \partial\{U_0 > 0\} \cap B_1) \geqslant \delta,$$

which is a contradiction with (8.21). The proof of Lemma 8.6 is thus finished. □

We are now in position to complete our analysis of the free boundary regularity results which follow from the study of the biharmonic measure by proving Theorem 1.10.

Proof of Theorem 1.10. We start by proving **1**°. For this, let $D \Subset \Omega$ and $x \in \mathcal{F}_\delta := \big(\partial\{u > 0\} \cap D\big) \setminus \mathcal{N}_\delta$, where $\mathcal{N}_\delta$ has been introduced before Lemma 8.6. Then there exists $r_x > 0$ such that

$$|\partial\{u > 0\} \cap B_{r_x}(x)| \leqslant C(n)\delta r_x^n,$$

where $C(n)$ is a dimensional constant. In this way, we can cover $\mathcal{F}_\delta$ with balls $B_{r_x}(x)$, and we can then extract a Besicovitch covering such that

$$|\mathcal{F}_\delta \cap D| \leqslant C(n)\delta\, |D|. \tag{8.22}$$

Then, sending $\delta \to 0$ the result in **1°** follows.

We now focus on **2°**. In this case, thanks to (1.18) we can use Lemma 8.6 and find a Besicovitch covering by balls $B_{r_x}(x)$ of $\mathcal{N}_\delta^{\text{sing}}$ such that

$$c_0(\delta) \sum r_x^{n-2} \leqslant \mathcal{M}_u(D') < \infty \tag{8.23}$$

where $D' \Supset D$ is a subdomain of Ω such that

$$\operatorname{dist}(D, \partial D') < \sup_{x \in \partial\{u>0\} \cap D} r_x := r_0.$$

Therefore, letting $r_0 \to 0$ in (8.23), we get that

$$\mathcal{H}^{n-2}(\mathcal{N}_\delta^{\text{sing}} \cap D) < +\infty. \tag{8.24}$$

Furthermore, since the free boundary is C^1 near points in $\mathcal{N}_\delta \setminus \mathcal{N}_\delta^{\text{sing}}$, we have that

$$\mathcal{H}^{n-2}\big((\mathcal{N}_\delta \setminus \mathcal{N}_\delta^{\text{sing}}) \cap D\big) < +\infty,$$

which, together with (8.24), implies that

$$\mathcal{H}^{n-2}(\mathcal{N}_\delta \cap D) < +\infty. \tag{8.25}$$

This gives the second claim in **2°**. We now prove the first claim in **2°**. For this, we use (8.22) and (8.25) to obtain that

$$|\partial\{u > 0\} \cap D| \leqslant |\mathcal{F}_\delta \cap D| + |\mathcal{N}_\delta \cap D| = |\mathcal{F}_\delta \cap D| \leqslant C(n)\delta\, |D|.$$

Then, sending $\delta \to 0$, we complete the proof of **2°**. ☐

Remark 8.7. If $\{u > 0\}$ is a John domain, then u is nondegenerate, due to the discussion in Subsection 8.2. Alternatively, as in Theorem 1.8, if $\{u > 0\}$ has uniformly positive Lebesgue density then u is nondegenerate.

We conclude this section by observing that, in general, $\{u = 0\}$ may have nonempty interior as the one-dimensional examples in Section 5 indicate. On the other hand, under the additional assumption that $\Delta u \leqslant 0$, we can show that $\partial\{u > 0\}$ and $\partial\{u < 0\}$ coincide, as stated in the following result:

Lemma 8.8. *Let u be a minimizer of the functional J defined in* (1.1)*. Assume that*

$$\Delta u \leqslant 0 \ \text{ in } \Omega. \tag{8.26}$$

Then, $\partial\{u > 0\}$ coincides with $\partial\{u < 0\}$.

Proof. From Lemma 4.1 we know that $f := \Delta u$ is super-harmonic in Ω, and therefore f is lower semicontinuous in Ω. Let us show that

$$\partial\{u > 0\} \subseteq \partial\{u < 0\}. \tag{8.27}$$

For this, assume by contradiction that $0 \in \big(\partial\{u > 0\}\big) \setminus \big(\partial\{u < 0\}\big)$. Then, there exists $\rho > 0$ such that $u \geqslant 0$ in B_ρ, with $\{u > 0\} \cap B_\rho \neq \varnothing$, and $\{u < 0\} \cap B_\rho \neq \varnothing$. In particular, u attains its minimum (equal to zero) inside B_ρ, which, combined with (8.26) and the Maximum Principle, implies that u vanishes identically. But this implies that $0 \notin \partial\{u > 0\}$, against the assumption. This proves (8.27). Now we show that

$$\partial\{u < 0\} \subseteq \partial\{u > 0\}. \tag{8.28}$$

To prove this, we argue by contradiction and suppose that $0 \in \big(\partial\{u < 0\}\big) \setminus \big(\partial\{u > 0\}\big)$. Then there exists $\rho > 0$ such that

$$u \leqslant 0 \qquad \text{in } B_\rho. \tag{8.29}$$

Since u cannot vanish identically, we can also assume that

$$\{u < 0\} \cap \partial B_\rho \neq \varnothing. \tag{8.30}$$

Let v be the harmonic function that coincides with u along ∂B_ρ. Then, by Maximum Principle, it follows from (8.29) and (8.30) that

$$v < 0 \text{ in } B_\rho, \tag{8.31}$$

and therefore

$$\int_{B_\rho} |\Delta v|^2 + \chi_{\{v>0\}} = 0.$$

The minimality of u thus implies that

$$\int_{B_\rho} |\Delta u|^2 + \chi_{\{u>0\}} = 0,$$

and so Δu must vanish in B_ρ and consequently u and v must coincide in B_ρ. This and (8.31) imply that 0 lies in the interior of $\{u < 0\}$, against the assumption, and this contradiction proves (8.28). Then, the desired result follows by combining (8.27) and (8.28). □

9. Stratification of free boundary, and proof of Theorem 1.12

In this section we reformulate some results obtained in Section 7 related to the dichotomy between the notion of rank-2 flatness and the quadratic growth of the minimizer.

In the setting of Definition 1.11, Theorem 1.7 can be reformulated as follows:

Proposition 9.1. *Let $u \in \mathcal{P}_r(\delta)$. Then there exist constants $C > 0$ and $r_0 > 0$, depending only on n, δ and r, such that*

$$|u(x)| \leqslant C|x|^2, \qquad \text{for any } x \in B_{r_0}.$$

Furthermore, recalling the definition of $h(r, x_0)$ in (1.14), a refinement of Theorem 1.7 can be formulated as follows:

Theorem 9.2. *Let $u \in \mathcal{P}_1$. Let $\delta \in (0, 1)$, $k > 10$ and $r_k := 2^{-k}$. Then, either $h(0, r_k) < \delta r_k$, or there exists a constant $C > 0$, depending only on n and δ, such that*

$$\sup_{B_{r_k/2}} |u| \leqslant C r_k^2.$$

We are now ready to complete the proof of Theorem 1.12.

Proof of Theorem 1.12. Notice that (1.19) and (1.20) follow as a consequence of Theorem 9.2. Therefore, to complete the proof of Theorem 1.12, it only remains to prove that u^+ is strongly nondegenerate at $z \in \mathcal{F}$. After rescaling $U_r(x) := r^{-2}u(z + rx)$, we see that it is enough to show that

$$\sup_{B_1} U_r^+ \geqslant \hat{C}, \tag{9.1}$$

for some $\hat{C} > 0$.

To check this, we first prove that

$$\begin{aligned} &\text{if } p \text{ is a homogeneous polynomial of degree two,} \\ &\text{then } \{p = 0\} \text{ is contained in the union of finitely many hypersurfaces.} \end{aligned} \tag{9.2}$$

Indeed, up to a linear transformation, and possibly exchanging the order of the variables, we can suppose that

$$p(x) = \sum_{i=1}^{n} a_i x_i^2,$$

with $(a_1, \ldots, a_m) \in \mathbb{R} \setminus \{0\}$ and $a_{m+1} = \cdots = a_n = 0$, for some $m \in \{1, \ldots, n\}$. Therefore the zero set of p is obtained by the zero set of the polynomial

$$\mathbb{R}^m \ni x \mapsto \tilde{p}(x) = \sum_{i=1}^{m} a_i x_i^2,$$

up to a Cartesian product with an $(n - m)$-dimensional linear space. Also,

$$\text{if } x \in \{\tilde{p} = 0\}, \text{ then } tx \in \{\tilde{p} = 0\} \text{ for all } t \in \mathbb{R}, \tag{9.3}$$

therefore

$$\{\tilde{p} = 0\} = \big\{tx,\ x \in \{\tilde{p} = 0\} \cap \mathbb{S}^{m-1}\big\}. \tag{9.4}$$

Furthermore

$$\{\nabla\tilde{p}=0\}=\{(2a_1x_1,\dots,2a_mx_m)=0\}=\{0\}. \tag{9.5}$$

Therefore, by (9.5), in the vicinity of any $x\in\{\tilde{p}=0\}\cap\mathbb{S}^{m-1}$, the set $\{\tilde{p}=0\}$ is an $(m-1)$-dimensional surface, which, in view of (9.3), is transverse to $\mathbb{S}^{m-1}$. Consequently, we have that $\{\tilde{p}=0\}\cap\mathbb{S}^{m-1}$ is the union of $(m-2)$-dimensional surfaces. In addition, from (9.5) we know that these surfaces cannot accumulate to each other, and so $\{\tilde{p}=0\}\cap\mathbb{S}^{m-1}$ is the union of finitely many $(m-2)$-dimensional surfaces. This and (9.4) imply that $\{\tilde{p}=0\}$ is the union of finitely many $(m-1)$-dimensional surfaces. Accordingly, we have that $\{p=0\}$ is the union of finitely many surfaces of dimension $(m-1)+(n-m)=n-1$. This completes the proof of (9.2).

Then, from (1.19) and (9.2), it follows that if $r=r_k$ is sufficiently small, then U_r satisfies the density estimate in (1.16). This allows us to exploit Theorem 1.8, from which we obtain (9.1), as desired. □

10. Monotonicity formula: proof of Theorem 1.13

This section is devoted to the proof of Theorem 1.13, which is based on a series of careful integration by parts aimed at spotting suitable integral cancellations. In addition, some "high order of differentiability" terms naturally appear in the computations, which need to be suitably removed in order to rigorously make sense of the formal manipulations. We start with some general computations valid in $\mathbb{R}^n$, then, from (10.24) on, we specialize to the case $n=2$. In this part of the paper, for the sake of shortness, we suppose that the assumptions of Theorem 1.13 are always satisfied without further mentioning them. Without loss of generality, we also suppose that $B_2\Subset\Omega$. Then, we have the following identity:

Lemma 10.1. *For every r_1, $r_2\in(0,3/2)$,*

$$4\int_{r_1}^{r_2}R(r)\,dr+2T(r_2)-2T(r_1)+D(r_2)-D(r_1)=0, \tag{10.1}$$

where

$$\begin{aligned}
R(r)\;&:=\frac{1}{r^{n+1}}\sum_{m=1}^{n}\int_{B_r}\Delta u\,\nabla u_m\cdot e_m-\sum_{m=1}^{n}\int_{\partial B_r}\Delta u\nabla u_m\cdot\frac{x^m\,x}{r^{n+2}}\\
&=\frac{1}{r^{n+1}}\int_{B_r}|\Delta u|^2-\frac{1}{r^n}\int_{\partial B_r}\Delta u\,\partial_r^2 u,\\
T(r)\;&:=\sum_{m=1}^{n}\int_{\partial B_r}\Delta u\,u_m\,\frac{x^m}{r^{n+1}}\\
&=\frac{1}{r^n}\int_{\partial B_r}\Delta u\,\partial_r u\\
\text{and}\qquad D(r)\;&:=\frac{1}{r^n}\int_{B_r}\big(|\Delta u|^2+\chi_{\{u>0\}}\big),
\end{aligned}\tag{10.2}$$

and the notation $\partial_r:=\frac{x}{|x|}\cdot\nabla$ has been used.

Proof. Fix $r\in(0,3/2)$. We let $\delta>0$ (to be taken as small as we wish in what follows), and consider a smooth function $\eta=\eta_\delta$ supported in $B_{r+\delta}$. Fixed $\varepsilon>0$, we also consider the mollifier $\rho_\varepsilon(x):=\frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$, for a given even function $\rho\in C_0^\infty(B_1)$. We also define $\phi=(\phi^1,\dots,\phi^n):\mathbb{R}^n\to\mathbb{R}^n$ as

$$\begin{aligned}
\mathbb{R}^n\ni x=(x^1,\dots,x^n)\longmapsto\phi^m(x)\quad&:=\quad(\psi^m*\rho_\varepsilon)(x),\\
\text{where}\qquad\psi^m(x)\quad&:=\quad x^m\eta(x).
\end{aligned}$$

Let also

$$F^m(x):=\Delta u(x)\,u_m(x). \tag{10.3}$$

In view of (4.1) and (4.2) (if u is a minimizer), or recalling that u is assumed to be in $C^{1,1}(\Omega)$ (if u is a one-phase minimizer), we know that

$$F^m\in L^p(B_1)\qquad\text{for every }p\in(1,+\infty).$$

We observe that ψ^m is supported in $B_{r+\delta}$ and so ϕ^m is supported in $B_{r+\delta+\varepsilon} \subset B_1$, as long as δ and ε are sufficiently small. Consequently,

$$\begin{aligned}
\int_\Omega \Delta u\, u_m\, \Delta\phi^m &= \int_{\mathbb{R}^n} \Delta u\, u_m\, \Delta\phi^m \\
&= \int_{\mathbb{R}^n} F^m \left(\Delta\psi^m * \rho_\varepsilon\right) = \iint_{\mathbb{R}^n\times B_\varepsilon(x)} F^m(x)\,\Delta\psi^m(y)\,\rho_\varepsilon(x-y)\,dx\,dy \\
&= \iint_{B_\varepsilon(x)\times\mathbb{R}^n} F^m(x)\,\Delta\psi^m(y)\,\rho_\varepsilon(y-x)\,dx\,dy = \iint_{\mathbb{R}^n} (F^m * \rho_\varepsilon)(y)\,\Delta\psi^m(y)\,dy \\
&= \int_\Omega F^m_\varepsilon\,\Delta\psi^m = -\int_\Omega \nabla F^m_\varepsilon\cdot\nabla\psi^m,
\end{aligned} \tag{10.4}$$

with

$$F^m_\varepsilon := F^m * \rho_\varepsilon. \tag{10.5}$$

Similarly, we have that

$$\int_\Omega \Delta u\nabla u_m\cdot\nabla\phi^m = \int_\Omega \Delta u\nabla u_m\cdot(\nabla\psi^m*\rho_\varepsilon) = \int_\Omega \big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot\nabla\psi^m. \tag{10.6}$$

Also,

$$\int_\Omega \Big(|\Delta u|^2+\chi_{\{u>0\}}\Big)\mathrm{div}\phi = \sum_{m=1}^n\int_\Omega \Big(|\Delta u|^2+\chi_{\{u>0\}}\Big)(\psi^m_m*\rho_\varepsilon) = \int_\Omega \Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big)\,\mathrm{div}\psi.$$

Then, we plug this information, (10.4) and (10.6) into (4.4) and we see that

$$\begin{aligned}
0 &= 2\int_\Omega \Delta u\sum_{m=1}^n \Big(2\nabla u_m\cdot\nabla\phi^m + u_m\Delta\phi^m\Big) - \int_\Omega \Big(|\Delta u|^2+\chi_{\{u>0\}}\Big)\mathrm{div}\phi \\
&= 4\sum_{m=1}^n\int_\Omega \big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot\nabla\psi^m - 2\sum_{m=1}^n\int_\Omega \nabla F^m_\varepsilon\cdot\nabla\psi^m - \int_\Omega \Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big)\,\mathrm{div}\psi.
\end{aligned} \tag{10.7}$$

Since the latter identity only involves the first derivatives of ψ^m, up to an approximation argument we can choose η to be the radial Lipschitz function defined by

$$\eta(x) := \begin{cases} 1 & \text{if } x\in B_r, \\ \dfrac{r+\delta-|x|}{\delta} & \text{if } x\in B_{r+\delta}\setminus B_r, \\ 0 & \text{if } x\in\mathbb{R}^n\setminus B_{r+\delta}. \end{cases}$$

In this way, we have that

$$\nabla\eta(x) = -\frac{x}{\delta\,|x|}\,\chi_{B_{r+\delta}\setminus B_r}(x)$$

$$\text{and}\qquad \nabla\psi^m(x) = e_m\eta(x) - \frac{x^m\,x}{\delta\,|x|}\,\chi_{B_{r+\delta}\setminus B_r}(x),$$

which also gives that

$$\mathrm{div}\psi(x) = n\eta(x) - \frac{|x|}{\delta}\,\chi_{B_{r+\delta}\setminus B_r}(x).$$

Therefore, we infer from (10.7) that

$$\begin{aligned}
0 &= 2\sum_{m=1}^n\int_{B_r}\Big(2\big((\Delta u\nabla u_m)*\rho_\varepsilon\big) - \nabla F^m_\varepsilon\Big)\cdot e_m - n\int_{B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big) \\
&\quad + 2\sum_{m=1}^n\int_{B_{r+\delta}\setminus B_r}\Big(2\big((\Delta u\nabla u_m)*\rho_\varepsilon\big) - \nabla F^m_\varepsilon\Big)\cdot\left(e_m\eta(x) - \frac{x^m\,x}{\delta\,|x|}\right) \\
&\quad - \int_{B_{r+\delta}\setminus B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big)\left(n\eta(x) - \frac{|x|}{\delta}\right).
\end{aligned}$$

Then, sending $\delta\to 0^+$, we deduce that

$$\begin{aligned}0 &= 2\sum_{m=1}^n\int_{B_r}\Big(2\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)-\nabla F^m_\varepsilon\Big)\cdot e_m - n\int_{B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big)\\ &\quad -2\sum_{m=1}^n\int_{\partial B_r}\Big(2\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)-\nabla F^m_\varepsilon\Big)\cdot\frac{x^m\,x}{r}\\ &\quad + r\int_{\partial B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big)\\ &= 2\sum_{m=1}^n\int_{B_r}G^m_\varepsilon\cdot e_m - n\int_{B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big)\\ &\quad -2\sum_{m=1}^n\int_{\partial B_r}G^m_\varepsilon\cdot\frac{x^m\,x}{r}+r\int_{\partial B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big),\end{aligned}\tag{10.8}$$

where

$$G^m_\varepsilon := 2\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)-\nabla F^m_\varepsilon.\tag{10.9}$$

Furthermore, letting

$$D_\varepsilon(r):=\frac{1}{r^n}\int_{B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big),\tag{10.10}$$

we have that

$$D'_\varepsilon(r)=\frac{1}{r^n}\int_{\partial B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big)-\frac{n}{r^{n+1}}\int_{B_r}\Big(\big(|\Delta u|^2+\chi_{\{u>0\}}\big)*\rho_\varepsilon\Big).\tag{10.11}$$

Thus, we multiply (10.8) by $\frac{1}{r^{n+1}}$ and we exploit (10.11) to conclude that

$$\begin{aligned}0 &= \frac{2}{r^{n+1}}\sum_{m=1}^n\int_{B_r}G^m_\varepsilon\cdot e_m-2\sum_{m=1}^n\int_{\partial B_r}G^m_\varepsilon\cdot\frac{x^m\,x}{r^{n+2}}+D'_\varepsilon(r)\\ &= 2Z_\varepsilon(r)+D'_\varepsilon(r),\end{aligned}\tag{10.12}$$

where

$$Z_\varepsilon(r):=\frac{1}{r^{n+1}}\sum_{m=1}^n\int_{B_r}G^m_\varepsilon\cdot e_m-\sum_{m=1}^n\int_{\partial B_r}G^m_\varepsilon\cdot\frac{x^m\,x}{r^{n+2}}.\tag{10.13}$$

Now, in light of (10.3), we observe that ∇F_m (and thus ∇F^ε_m) involves third derivatives, and therefore we aim at "lowering the order of derivative" of this term from (10.13) in view of (10.9) (and this goal will be accomplished via a suitable averaging procedure). To this end, we observe that

$$\int_{B_r}\nabla F^m_\varepsilon\cdot e_m=\int_{B_r}\operatorname{div}(F^m_\varepsilon e_m)=\int_{\partial B_r}F^m_\varepsilon e_m\cdot\frac{x}{r}=\int_{\partial B_r}F^m_\varepsilon\,\frac{x^m}{r}.\tag{10.14}$$

We notice that the last term in (10.14) does not contain any third order derivatives. As for the boundary term in (10.8) that involves the third derivative, we have that

$$\begin{aligned}\int_{\partial B_r}\nabla F^m_\varepsilon\cdot\frac{x^m\,x}{r^{n+2}} &= \int_{\partial B_1}\nabla F^m_\varepsilon(rx)\cdot\frac{x^m\,x}{r}\\ &= \int_{\partial B_1}\partial_r(F^m_\varepsilon(rx))\cdot\frac{x^m}{r}\\ &= \frac{d}{dr}\left\{\int_{\partial B_1}F^m_\varepsilon(rx)\frac{x^m}{r}\right\}+\int_{\partial B_1}F^m_\varepsilon(rx)\frac{x^m}{r^2}\\ &= \frac{d}{dr}\left\{\int_{\partial B_r}F^m_\varepsilon\frac{x^m}{r^{n+1}}\right\}+\int_{\partial B_r}F^m_\varepsilon\frac{x^m}{r^{n+2}}.\end{aligned}$$

As a consequence, using the latter identity, (10.9) and (10.14), we find that

$$\begin{aligned}\int_{B_r} G^m_\varepsilon\cdot e_m &= 2\int_{B_r}\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot e_m-\int_{B_r}\nabla F^m_\varepsilon\cdot e_m\\ &= 2\int_{B_r}\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot e_m-\int_{\partial B_r}F^m_\varepsilon\frac{x^m}{r}\end{aligned}$$

and

$$\begin{aligned}\int_{\partial B_r} G^m_\varepsilon\cdot \frac{x^m\,x}{r^{n+2}} &= 2\int_{\partial B_r}\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot \frac{x^m\,x}{r^{n+2}}-\int_{\partial B_r}\nabla F^m_\varepsilon\cdot \frac{x^m\,x}{r^{n+2}}\\ &= 2\int_{\partial B_r}\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot \frac{x^m\,x}{r^{n+2}}-\int_{\partial B_r}F^m_\varepsilon\frac{x^m}{r^{n+2}}-\frac{d}{dr}\left\{\int_{\partial B_r}F^m_\varepsilon\frac{x^m}{r^{n+1}}\right\}.\end{aligned}$$

From this and (10.13), we obtain that

$$\begin{aligned}Z_\varepsilon(r) &= \frac{2}{r^{n+1}}\sum_{m=1}^n\int_{B_r}\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot e_m-2\sum_{m=1}^n\int_{\partial B_r}\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot\frac{x^m\,x}{r^{n+2}}+\sum_{m=1}^n\frac{d}{dr}\left\{\int_{\partial B_r}F^m_\varepsilon\frac{x^m}{r^{n+1}}\right\}\\ &= 2R_\varepsilon(r)+T'_\varepsilon(r),\end{aligned}$$

with

$$\begin{aligned}&R_\varepsilon(r):=\frac{1}{r^{n+1}}\sum_{m=1}^n\int_{B_r}\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot e_m-\sum_{m=1}^n\int_{\partial B_r}\big((\Delta u\nabla u_m)*\rho_\varepsilon\big)\cdot\frac{x^m\,x}{r^{n+2}}\\ &\text{and}\;\; T_\varepsilon(r):=\sum_{m=1}^n\left\{\int_{\partial B_r}F^m_\varepsilon\frac{x^m}{r^{n+1}}\right\}=\sum_{m=1}^n\left\{\int_{\partial B_r}(\Delta u\,u_m)*\rho_\varepsilon\frac{x^m}{r^{n+1}}\right\},\end{aligned}\tag{10.15}$$

where we have also used (10.3) and (10.5).

Consequently, integrating (10.12),

$$\begin{aligned}0 &= 2\int_{r_1}^{r_2}Z_\varepsilon(r)\,dr+D_\varepsilon(r_2)-D_\varepsilon(r_1)\\ &= 4\int_{r_1}^{r_2}R_\varepsilon(r)\,dr+2T_\varepsilon(r_2)-2T_\varepsilon(r_1)+D_\varepsilon(r_2)-D_\varepsilon(r_1).\end{aligned}\tag{10.16}$$

Comparing (10.2) with (10.15), we see that $R_\varepsilon\to R$ and $T_\varepsilon\to T$ as $\varepsilon\to0$, thanks to (4.1) and (4.2).

We thereby obtain the desired claim in (10.1) by passing to the limit the identity in (10.16). ☐

We also point out the following useful calculation:

Lemma 10.2. *In the notation stated by (10.2), we have that*

$$4\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta u\left(2\frac{u_r}{r}-\partial_r^2u-2\frac{u}{r^2}\right)\right)dr-4V(r_2)+4V(r_1)+2T(r_2)-2T(r_1)+D(r_2)-D(r_1)=0,\tag{10.17}$$

where

$$V(r):=\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta u u.\tag{10.18}$$

Proof. For any smooth function v,

$$\begin{aligned}\int_{B_r}|\Delta v|^2 &= \int_{B_r}\Big(\mathrm{div}(\Delta v\nabla v)-\nabla\Delta v\cdot\nabla v\Big)=\int_{\partial B_r}\Delta v\,v_r-\int_{B_r}\nabla\Delta v\cdot\nabla v\\ &= \int_{\partial B_r}\Delta v\,v_r-\int_{B_r}\mathrm{div}(v\nabla\Delta v)+\int_{B_r}\Delta^2v\,v\\ &= \int_{\partial B_r}\Delta v\,v_r-\int_{\partial B_r}v\,\Delta v_r+\int_{B_r}\Delta^2v\,v.\end{aligned}\tag{10.19}$$

We also observe that

$$\frac{d}{dr}\left(\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta vv\right)$$

$$
\begin{aligned}
&= \frac{d}{dr}\left(\frac{1}{r^2}\int_{\partial B_1}\Delta v(r\theta)v(r\theta)\right)\\
&= -\frac{2}{r^3}\int_{\partial B_1}\Delta v(r\theta)v(r\theta)+\frac{1}{r^2}\int_{\partial B_1}\Delta v_r(r\theta)v(r\theta)+\frac{1}{r^2}\int_{\partial B_1}\Delta v(r\theta)v_r(r\theta)\\
&= -\frac{2}{r^{n+2}}\int_{\partial B_r}\Delta v\, v+\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v_r\, v+\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v\, v_r.
\end{aligned}
$$

From this and (10.19), we obtain that, for any smooth function v,

$$
\begin{aligned}
&\frac{1}{r^{n+1}}\int_{B_r}|\Delta v|^2-\frac{1}{r^n}\int_{\partial B_r}\Delta v\,\partial_r^2 v\\
=\;& \frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v\, v_r-\frac{1}{r^{n+1}}\int_{\partial B_r}v\,\Delta v_r+\frac{1}{r^{n+1}}\int_{B_r}\Delta^2 v\, v-\frac{1}{r^n}\int_{\partial B_r}\Delta v\,\partial_r^2 v\\
=\;& \frac{1}{r^n}\int_{\partial B_r}\Delta v\left(2\frac{v_r}{r}-\partial_r^2 v-2\frac{v}{r^2}\right)+\frac{1}{r^{n+1}}\int_{B_r}\Delta^2 v\, v-\frac{d}{dr}\left(\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v v\right).
\end{aligned}
$$

Integrating this identity and setting

$$
V_v(r):=\frac{1}{r^{n+1}}\int_{\partial B_r}\Delta v v, \tag{10.20}
$$

we thereby obtain that

$$
\begin{aligned}
&\int_{r_1}^{r_2}\left(\frac{1}{r^{n+1}}\int_{B_r}|\Delta v|^2-\frac{1}{r^n}\int_{\partial B_r}\Delta v\,\partial_r^2 v\right)dr\\
=\;&\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta v\left(2\frac{v_r}{r}-\partial_r^2 v-2\frac{v}{r^2}\right)+\frac{1}{r^{n+1}}\int_{B_r}\Delta^2 v\, v\right)dr-V_v(r_2)+V_v(r_1).
\end{aligned} \tag{10.21}
$$

The idea is now to take v as a mollification of u, and use either (8.4) (if u is a minimizer) or Lemma 8.5 (if u is a one-phase minimizer). In this way, the term

$$
\int_{B_r}\Delta^2 v\, v
$$

approaches either

$$
\int_{B_r}u\,\mathcal{M}_u,
$$

in the notation of (8.4) (if u is a minimizer), or 0 (if u is a one-phase minimizer, due to Lemma 8.5).

To make the notation uniform, we therefore define $\mathcal{M}_u^*:=\mathcal{M}_u$ if u is a minimizer and $\mathcal{M}_u^*:=0$ if u is a one-phase minimizer: then, approximating u, passing to the limit (10.21) and comparing (10.20) with (10.18), we can write

$$
\begin{aligned}
&\int_{r_1}^{r_2}\left(\frac{1}{r^{n+1}}\int_{B_r}|\Delta u|^2-\frac{1}{r^n}\int_{\partial B_r}\Delta u\,\partial_r^2 u\right)dr\\
=\;&\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta u\left(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\right)-\frac{1}{r^{n+1}}\int_{B_r}u\,\mathcal{M}_u^*\right)dr-V(r_2)+V(r_1).
\end{aligned}
$$

That is, recalling (10.2),

$$
\int_{r_1}^{r_2}R(r)\,dr=\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta u\left(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\right)-\frac{1}{r^{n+1}}\int_{B_r}u\,\mathcal{M}_u^*\right)dr-V(r_2)+V(r_1).
$$

From this and (10.1) we obtain that

$$
\begin{aligned}
&2T(r_1)-2T(r_2)+D(r_1)-D(r_2)\\
&=4\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r}\Delta u\left(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\right)-\frac{1}{r^{n+1}}\int_{B_r}u\,\mathcal{M}_u^*\right)dr-4V(r_2)+4V(r_1).
\end{aligned} \tag{10.22}
$$

Now we claim that

$$
\int_{B_r}u\,\mathcal{M}_u^*=0. \tag{10.23}
$$

For this, since $\mathcal{M}^*_u = 0$ in the one-phase problem, we can suppose that u is a minimizer, in which case $\mathcal{M}^*_u = \mathcal{M}_u$. Then, let us fix $\delta \in (0,1)$. From Lemma 4.1, we know that

$$-\int_{B_r\cap\{|u|\geqslant\delta\}} u\,\mathcal{M}_u = \int_{B_r\cap\{u\geqslant\delta\}} u\,\Delta^2 u + \int_{B_r\cap\{u\leqslant-\delta\}} u\,\Delta^2 u = 0.$$

Therefore, exploiting Lemma 8.4,

$$\left|\int_{B_r} u\,\mathcal{M}_u\right| = \left|\int_{B_r\cap\{|u|<\delta\}} u\,\mathcal{M}_u\right| \leqslant \delta\mathcal{M}_u(B_r) \leqslant C\delta r^{n-2},$$

for some $C>0$. Then, sending $\delta\to 0^+$, we obtain (10.23) as desired.

Then, the identities in (10.22) and (10.23) lead to (10.17). □

Now we restrict the previous calculations to the case $n=2$, and we complete the proof of (1.21).

Proof of (1.21). Using using polar coordinates (r,θ), we compute

$$\begin{aligned}
-\frac{1}{r^n}\int_{\partial B_r}\Delta u\left(2\frac{u_r}{r}-\partial_r^2 u-2\frac{u}{r^2}\right) &= \int_{\partial B_1}\frac{1}{r}\Delta u\left(u_{rr}-2\frac{u_r}{r}+2\frac{u}{r^2}\right)\\
&= \int_{\partial B_1}\frac{1}{r}\left(u_{rr}+\frac{u_r}{r}+\frac{u_{\theta\theta}}{r^2}\right)\left(u_{rr}-2\frac{u_r}{r}+2\frac{u}{r^2}\right)\\
&= A(r)+B(r),
\end{aligned}\tag{10.24}$$

where

$$\begin{aligned}
&A(r) := \int_{\partial B_1}\frac{1}{r^3}u_{\theta\theta}\left(u_{rr}-2\frac{u_r}{r}+2\frac{u}{r^2}\right)\\
\text{and}\quad &B(r) := \int_{\partial B_1}\frac{1}{r}\left(u_{rr}+\frac{u_r}{r}\right)\left(u_{rr}-2\frac{u_r}{r}+2\frac{u}{r^2}\right).
\end{aligned}\tag{10.25}$$

Now we perform several integrations by parts that involve the terms related to $A(r)$. First of all, we see that

$$\begin{aligned}
\frac{1}{r^3}\int_{\partial B_1}u_{\theta\theta}u_{rr} &= -\frac{1}{r^3}\int_{\partial B_1}u_\theta u_{\theta rr}\\
&= -\frac{d}{dr}\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}+\int_{\partial B_1}\frac{u_{r\theta}^2}{r^3}-3\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}.
\end{aligned}\tag{10.26}$$

Similarly, we have that

$$-2\int_{\partial B_1}\frac{1}{r^4}u_{\theta\theta}u_r = 2\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4} = 2\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}\tag{10.27}$$

and

$$2\int_{\partial B_1}\frac{1}{r^5}u_{\theta\theta}u = -2\int_{\partial B_1}\frac{u_\theta^2}{r^5}.\tag{10.28}$$

Combining (10.26), (10.27) and (10.28), and recalling (10.25), we get

$$\begin{aligned}
A(r) &= -\frac{d}{dr}\left(\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}\right)+\int_{\partial B_1}\frac{u_{r\theta}^2}{r^3}-3\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}+2\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}-2\int_{\partial B_1}\frac{u_\theta^2}{r^5}\\
&= -\frac{d}{dr}\left(\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}\right)+\int_{\partial B_1}\frac{u_{r\theta}^2}{r^3}-\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}-2\int_{\partial B_1}\frac{u_\theta^2}{r^5}\\
&= -\frac{d}{dr}\left(\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}\right)+\int_{\partial B_1}\frac{1}{r^3}\left(u_{\theta r}-\frac{2u_\theta}{r}\right)^2+3\int_{\partial B_1}\frac{u_\theta u_{\theta r}}{r^4}-6\int_{\partial B_1}\frac{u_\theta^2}{r^5}\\
&= -\frac{d}{dr}\left(\int_{\partial B_1}\frac{u_\theta u_{r\theta}}{r^3}+\frac{3}{2}\int_{\partial B_1}\frac{u_\theta^2}{r^4}\right)+\int_{\partial B_1}\frac{1}{r^3}\left(u_{\theta r}-\frac{2u_r}{r}\right)^2\\
&= -\frac{d}{dr}\left(\int_{\partial B_r}\frac{u_\theta u_{r\theta}}{r^4}+\frac{3}{2}\int_{\partial B_r}\frac{u_\theta^2}{r^5}\right)+\int_{\partial B_r}\frac{1}{r^4}\left(u_{\theta r}-\frac{2u_r}{r}\right)^2.
\end{aligned}\tag{10.29}$$

From (10.25), we also compute that

$$
\begin{aligned}
B(r) &= \int_{\partial B_1} \frac{1}{r}\left(u_{rr}^2 - \frac{2u_{rr}u_r}{r} + \frac{2uu_{rr}}{r^2} + \frac{u_r u_{rr}}{r} - \frac{2u_r^2}{r^2} + \frac{2uu_r}{r^3}\right)\\
&= \int_{\partial B_1} \frac{1}{r}\left(u_{rr}^2 - \frac{u_{rr}u_r}{r} + \frac{2uu_{rr}}{r^2} - \frac{2u_r^2}{r^2} + \frac{2uu_r}{r^3}\right)\\
&= \int_{\partial B_1} \frac{1}{r}\left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2 + \frac{1}{r}\left(\frac{5u_r u_{rr}}{r} - \frac{6uu_{rr}}{r^2} - \frac{11u_r^2}{r^2} + \frac{26uu_r}{r^3} - \frac{16u^2}{r^4}\right)\\
&= \int_{\partial B_1} \frac{1}{r}\left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2 + \frac{d}{dr}\left(\int_{\partial B_1} \frac{5u_r^2}{2r^2} - \int_{\partial B_1} \frac{6uu_r}{r^3} + \int_{\partial B_1} \frac{4u^2}{r^4}\right)\\
&= \int_{\partial B_r} \frac{1}{r^2}\left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2 + \frac{d}{dr}\left(\int_{\partial B_r} \frac{5u_r^2}{2r^3} - \int_{\partial B_r} \frac{6uu_r}{r^4} + \int_{\partial B_r} \frac{4u^2}{r^5}\right).
\end{aligned} \tag{10.30}
$$

Using (10.29) and (10.30), we conclude that

$$
A(r) + B(r) = \frac{1}{r^2}\int_{\partial B_r}\left[\left(\frac{u_{\theta r}}{r} - \frac{2u_r}{r^2}\right)^2 + \left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2\right] + W'(r), \tag{10.31}
$$

where

$$
W(r) := \int_{\partial B_r}\left(\frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_\theta u_{r\theta}}{r^4} - \frac{3u_\theta^2}{2r^5}\right). \tag{10.32}
$$

Now, from (10.17) and (10.24), we see that

$$
\begin{aligned}
&-4V(r_2) + 4V(r_1) + 2T(r_2) - 2T(r_1) + D(r_2) - D(r_1)\\
= \;& -4\int_{r_1}^{r_2}\left(\frac{1}{r^n}\int_{\partial B_r} \Delta u\left(2\frac{u_r}{r} - \partial_r^2 u - 2\frac{u}{r^2}\right)\right) dr\\
= \;& 4\int_{r_1}^{r_2}\big(A(r) + B(r)\big)\, dr.
\end{aligned}
$$

This and (10.31) give that

$$
\begin{aligned}
&- V(r_2) + V(r_1) + \frac{T(r_2) - T(r_1)}{2} + \frac{D(r_2) - D(r_1)}{4} - W(r_2) + W(r_1)\\
&= \int_{r_1}^{r_2}\left\{\frac{1}{r^2}\int_{\partial B_r}\left[\left(\frac{u_{\theta r}}{r} - \frac{2u_r}{r^2}\right)^2 + \left(u_{rr} - \frac{3u_r}{r} + 4\frac{u}{r^2}\right)^2\right]\right\}.
\end{aligned} \tag{10.33}
$$

Recalling (1.22), (10.2), (10.18) and (10.32), we see that

$$
\begin{aligned}
&-V(r) + \frac{T(r)}{2} + \frac{D(r)}{4} - \int_{\partial B_r}\left(\frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_\theta u_{r\theta}}{r^4} - \frac{3u_\theta^2}{2r^5}\right)\\
= \;& -\frac{1}{r^3}\int_{\partial B_r} \Delta u u + \frac{1}{2r^2}\int_{\partial B_r} \Delta u\, \partial_r u + \frac{1}{4r^2}\int_{B_r}\left(|\Delta u|^2 + \chi_{\{u>0\}}\right)\\
&\quad - \int_{\partial B_r}\left(\frac{5u_r^2}{2r^3} - \frac{6uu_r}{r^4} + \frac{4u^2}{r^5} - \frac{u_\theta u_{r\theta}}{r^4} - \frac{3u_\theta^2}{2r^5}\right)\\
= \;& E(r).
\end{aligned}
$$

This and (10.33) establish (1.21), as desired. $\square$

Now, since the proof of (1.21) has been completed, to finish the proof of Theorem 1.13, we only need to show that the function E defined in (1.22) is bounded and to check that if E is constant then u is a homogeneous function of degree two.

These goals will be accomplished by the following arguments:

Proof of the boundedness of E. To show that E is bounded, we claim that the exist $C > 0$ and a sequence $r_k \to 0^+$ such that

$$\int_{\partial B_{r_k}} \left(\frac{|\nabla u|^2}{r_k^3} + \frac{|D^2 u|^2}{r_k} \right) \leqslant C. \tag{10.34}$$

The proof of (10.34) needs to distinguish the case in which u is a minimizer from the case in which u is a one-phase minimizer. Suppose first that u is a one-phase minimizer. Then, since $u(0) = 0 \leqslant u(x)$ for any $x \in \Omega$ and u is assumed to be $C^{1,1}(\Omega)$, we can write that $|\nabla u(x)| \leqslant C|x|$ and $|D^2 u(x)| \leqslant C$, for some $C > 0$, from which (10.34) plainly follows in this case.

Now, we prove (10.34) assuming that u is a minimizer We argue by contradiction, supposing that (10.34) does not hold. Then, for any $\bar{C} > 0$ there exists $\bar{r} \in (0, 1)$ such that for any $r \in (0, \bar{r})$ we have that

$$\int_{\partial B_r} \left(\frac{|\nabla u|^2}{r^3} + \frac{|D^2 u|^2}{r} \right) \geqslant \bar{C}.$$

This, Corollary A.2 (if u is a minimizer) or the fact that u is assumed to be in $C^{1,1}(\Omega)$ (if u is a one-phase minimizer) lead that, for a suitable $C > 0$,

$$\begin{aligned}
C &\geqslant \frac{1}{\bar{r}^4} \int_{B_{\bar{r}}} |\nabla u|^2 + \frac{1}{\bar{r}^2} \int_{B_{\bar{r}}} |D^2 u|^2 \\
&= \frac{1}{\bar{r}^4} \int_0^{\bar{r}} \left(\int_{\partial B_r} |\nabla u|^2 \right) dr + \frac{1}{\bar{r}^2} \int_0^{\bar{r}} \left(\int_{\partial B_r} |D^2 u|^2 \right) dr \\
&= \frac{1}{\bar{r}} \int_0^{\bar{r}} \left(\int_{\partial B_r} \frac{|\nabla u|^2}{\bar{r}^3} + \int_{\partial B_r} \frac{|D^2 u|^2}{\bar{r}} \right) dr \\
&\geqslant \frac{1}{\bar{r}} \int_{\frac{\bar{r}}{2}}^{\bar{r}} \left(\int_{\partial B_r} \frac{|\nabla u|^2}{\bar{r}^3} + \int_{\partial B_r} \frac{|D^2 u|^2}{\bar{r}} \right) dr \\
&\geqslant \frac{1}{8\bar{r}} \int_{\frac{\bar{r}}{2}}^{\bar{r}} \left(\int_{\partial B_r} \frac{|\nabla u|^2}{r^3} + \int_{\partial B_r} \frac{|D^2 u|^2}{r} \right) dr \\
&\geqslant \frac{\bar{C}}{16},
\end{aligned}$$

which is a contradiction if $\bar{C}$ is suitably large, and this establishes (10.34).

As a consequence, using the Cauchy-Schwarz inequality, Theorem 1.7 and (10.34),

$$\begin{aligned}
&\int_{\partial B_{r_k}} \left| \frac{\Delta u\, u_r}{2r_k^2} - \frac{5u_r^2}{2r_k^3} - \frac{\Delta u u}{r_k^3} + \frac{6uu_r}{r_k^4} + \frac{u_\theta u_{\theta r}}{r_k^4} - \frac{4u^2}{r_k^5} - \frac{3u_\theta^2}{2r_k^5} \right| \\
\leqslant\ & C \int_{\partial B_{r_k}} \left(\frac{|D^2 u|\, |\nabla u|}{r_k^{\frac{1}{2}}\, r_k^{\frac{3}{2}}} + \frac{|\nabla u|^2}{r_k^3} + \frac{|\Delta u|}{r_k^{\frac{1}{2}}\, r_k^{\frac{1}{2}}} + \frac{|\nabla u|}{r_k^{\frac{3}{2}}\, r_k^{\frac{1}{2}}} + \frac{1}{r_k} \right) \\
\leqslant\ & C \int_{\partial B_{r_k}} \left(\frac{|\nabla u|^2}{r_k^3} + \frac{|D^2 u|^2}{r_k} + \frac{1}{r_k} \right) \\
\leqslant\ & C,
\end{aligned}$$

for some $C > 0$, possibly varying from line to line.

Using this, (1.22) and Corollary A.2 (if u is a minimizer) or the assumption that $u \in C^{1,1}(\Omega)$ (if u is a one-phase minimizer), we thereby deduce that

$$\begin{aligned}
&|E(r_k)| \\
&\leqslant \int_{\partial B_{r_k}} \left| \frac{\Delta u\, u_r}{2r_k^2} - \frac{5u_r^2}{2r_k^3} - \frac{\Delta u u}{r_k^3} + \frac{6uu_r}{r_k^4} + \frac{u_\theta u_{\theta r}}{r_k^4} - \frac{4u^2}{r_k^5} - \frac{3u_\theta^2}{2r_k^5} \right| + \frac{1}{4r_k^2} \int_{B_{r_k}} \left(|\Delta u|^2 + \chi_{\{u>0\}} \right) \\
&\leqslant C + \frac{1}{4r_k^2} \int_{B_{r_k}} \chi_{\{u>0\}} \\
&\leqslant C,
\end{aligned} \tag{10.35}$$

up to renaming $C > 0$.

Now, fix $r \in (0,1)$. Let $\bar{k}$ sufficiently large, such that $r_{\bar{k}} \in (0,r)$. From (1.21), we know that

$$E(r_{\bar{k}}) \leqslant E(r) \leqslant E(1).$$

Hence, by (10.35),

$$-C \leqslant E(r) \leqslant E(1),$$

and this shows that E is bounded, as desired. □

Having already checked the validity of the monotonicity formula in (1.21) and the fact that E is bounded, in order to complete the proof of Theorem 1.13, we only need to show that if E is constant in $(0,\tau)$, then u is a homogeneous function of degree two. This is now a simple consequence of (1.21). The detailed argument goes as follows.

Proof of the case of constant E. Suppose now that E is constant in $(0,\tau)$. Then, by (1.21),

$$\begin{aligned} &-\frac{\partial}{\partial\theta}\left(-\frac{u_r}{r}+\frac{2u}{r^2}\right)=\frac{u_{r\theta}}{r^2}-\frac{2u_\theta}{r}=0 \\ \text{and} \qquad &-r\frac{\partial}{\partial r}\left(-\frac{u_r}{r}+\frac{2u}{r^2}\right)=u_{rr}-\frac{3u_r}{r}+\frac{4u}{r^2}=0, \end{aligned}$$

which, in turn, gives that

$$\nabla\left(-\frac{u_r}{r}+2\frac{u}{r^2}\right)=0.$$

Consequently, the function $-\frac{u_r}{r}+\frac{2u}{r^2}$ is constant for $|x| \in (0,\tau)$, hence we write

$$-\frac{u_r}{r}+\frac{2u}{r^2}=c, \tag{10.36}$$

for some $c \in \mathbb{R}$.

Now we define

$$v(r,\theta) := u(r,\theta)+cr^2\log r. \tag{10.37}$$

Using (10.36), we obtain that

$$v_r = u_r + 2cr\log r + cr = \frac{2u}{r} + 2cr\log r = \frac{2v}{r}.$$

Integrating this equation, fixed $\bar{r} \in (0,\tau)$, we find that

$$v(r,\theta) = \frac{r^2\, v(\bar{r},\theta)}{\bar{r}^2}.$$

This and (10.37) give that

$$u(r,\theta) = \frac{r^2\, v(\bar{r},\theta)}{\bar{r}^2} - cr^2\log r.$$

Therefore, exploiting Theorem 1.7 (if u is a minimizer) or the assumption that $u \in C^{1,1}(\Omega)$ (if u is a one-phase minimizer),

$$C \geqslant \frac{|u(r,\theta)|}{r^2} \geqslant |c|\,|\log r| - \frac{|v(\bar{r},\theta)|}{\bar{r}^2},$$

for some $C > 0$ and therefore

$$|c| \leqslant \lim_{r\to 0} \frac{|v(\bar{r},\theta)|}{\bar{r}^2\,|\log r|} + \frac{C}{|\log r|} = 0.$$

Hence, we get that $c = 0$ and, as a consequence, we can write (10.36) as

$$-\frac{u_r}{r}+\frac{2u}{r^2}=0$$

for any $x \in B_\tau$, and therefore $\nabla u(x)\cdot x = 2u(x)$ for any $x \in B_\tau$. Observing that this is the Euler equation for homogeneous functions of degree two, we thus obtain the homogeneity of u. The proof of Theorem 1.13 is thereby complete. □

We finish this section by an explicit computation of the energy E for the homogeneous functions of degree two on the plane. It will be used later in the proof of Theorem 1.15.

Lemma 10.3. *Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a cone in $\mathbb{R}^2$, written in polar coordinates as*

$$\mathcal{C} = \big\{(r,\theta) \in (0,+\infty) \times (\theta_1,\theta_2)\big\},$$

for some $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$.

Let $u : \mathcal{C} \to \mathbb{R}$ be a homogeneous function of the form $u(x) = r^2 g(\theta)$, with $g \in C^2([\theta_1,\theta_2])$, $g > 0$ in (θ_1,θ_2), and

$$g(\theta_1) = g(\theta_2) = 0 \qquad \text{and} \qquad g'(\theta_1) = g'(\theta_2) = 0.$$

Assume also that Δu is constant in $\mathcal{C}$. Then, for any $r > 0$,

$$\begin{aligned}\int_{\mathcal{C}\cap\partial B_r} &\left(\frac{\Delta u\, u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6uu_r}{r^4} + \frac{u_\theta u_{\theta r}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_\theta^2}{2r^5}\right) + \frac{1}{4r^2}\int_{\mathcal{C}\cap B_r} \big(|\Delta u|^2 + \chi_{\{u>0\}}\big)\\ &= \frac{\pi}{4}\,\frac{|\{u>0\}\cap B_r|}{|B_r|} = \frac{\theta_2-\theta_1}{8}.\end{aligned} \tag{10.38}$$

Proof. By assumption, in $\mathcal{C}$ we have that

$$C_0 = \Delta u = 4g + g'', \tag{10.39}$$

for some $C_0 \in \mathbb{R}$, and

$$\begin{aligned}&\frac{\Delta u\, u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6uu_r}{r^4} + \frac{u_\theta u_{\theta r}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_\theta^2}{2r^5}\\ =\;& \frac{(4g+g'')g}{r} - \frac{10g^2}{r} - \frac{(4g+g'')g}{r} + \frac{12g^2}{r} + \frac{2(g')^2}{r} - \frac{4g^2}{r} - \frac{3(g')^2}{2r}\\ =\;& -\frac{2g^2}{r} + \frac{(g')^2}{2r}.\end{aligned}$$

Therefore, after an integration by parts, and recalling (10.39), we have that

$$\begin{aligned}&\int_{\mathcal{C}\cap\partial B_r} \left(\frac{\Delta u\, u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6uu_r}{r^4} + \frac{u_\theta u_{\theta r}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_\theta^2}{2r^5}\right)\\ =\;& \int_{\theta_1}^{\theta_2} \left(-2g^2 + \frac{(g')^2}{2}\right)\\ =\;& \int_{\theta_1}^{\theta_2} \left(-2g^2 - \frac{g''g}{2}\right)\\ =\;& -\frac{1}{2}\int_{\theta_1}^{\theta_2} g(4g+g'')\\ =\;& -\frac{C_0}{2}\int_{\theta_1}^{\theta_2} g\\ =\;& \frac{C_0}{8}\int_{\theta_1}^{\theta_2} (g''-C_0)\\ =\;& -\frac{C_0^2\,(\theta_2-\theta_1)}{8}.\end{aligned} \tag{10.40}$$

On the other hand,

$$\frac{1}{4r^2}\int_{\mathcal{C}\cap B_r} |\Delta u|^2 = \frac{1}{8}\int_{\theta_1}^{\theta_2} (4g+g'')^2 = \frac{C_0^2\,(\theta_2-\theta_1)}{8}.$$

This and (10.40) give that

$$\begin{aligned}\int_{\mathcal{C}\cap\partial B_r} &\left(\frac{\Delta u\, u_r}{2r^2} - \frac{5u_r^2}{2r^3} - \frac{\Delta u u}{r^3} + \frac{6uu_r}{r^4} + \frac{u_\theta u_{\theta r}}{r^4} - \frac{4u^2}{r^5} - \frac{3u_\theta^2}{2r^5}\right) + \frac{1}{4r^2}\int_{\mathcal{C}\cap B_r} \big(|\Delta u|^2 + \chi_{\{u>0\}}\big)\\ &= \frac{1}{4r^2}\int_{B_r} \chi_{\{u>0\}},\end{aligned}$$

which proves (10.38). □

11. Monotonicity formula: homogeneity of the blow-up limits, and proof of Theorem 1.14

In this section, we apply the results in Theorem 1.13 to study the homogeneity properties of the blow-up limits of the minimizers of J at free boundary points with vanishing gradient, thus proving Theorem 1.14.

Proof of Theorem 1.14. Suppose that u does not vanish identically. We let

$$Q(u,x) := Q(u,r,\theta) = \Big(-\frac{u_{r\theta}}{r} + 2\frac{u_\theta}{r^2}\Big)^2 + \Big(u_{rr} - 3\frac{u_r}{r} + 4\frac{u}{r^2}\Big)^2. \tag{11.1}$$

Note that Q is invariant with respect to quadratic scaling. Indeed, if we define, for any $s > 0$,

$$u_s(x) := \frac{u(sx)}{s^2},$$

we have that

$$\begin{aligned} Q(u_s,x) &= \left(-\frac{(u_s)_{r\theta}}{r} + 2\frac{(u_s)_\theta}{r^2}\right)^2 + \left((u_s)_{rr} - 3\frac{(u_s)_r}{r} + 4\frac{u_s}{r^2}\right)^2 \\ &= \left(-\frac{u_{r\theta}(sx)}{sr} + 2\frac{u_\theta(sx)}{(sr)^2}\right)^2 + \left(u_{rr}(sx) - 3\frac{u_r(sx)}{sr} + 4\frac{u(sx)}{(sr)^2}\right)^2 \\ &= Q(u,sx). \end{aligned} \tag{11.2}$$

Now, in view of (1.21) and (11.1), we observe that

$$\begin{aligned} E(\tau_2) - E(\tau_1) &= \int_{\tau_1}^{\tau_2} \left\{\frac{1}{r^2}\int_{\partial B_r} \left[\left(\frac{u_{\theta r}}{r} - \frac{2u_\theta}{r^2}\right)^2 + \left(u_{rr} - \frac{3u_r}{r} + \frac{4u}{r^2}\right)^2\right]\right\} dr \\ &= \int_{\tau_1}^{\tau_2} \left(\frac{1}{r^2}\int_{\partial B_r} Q(u,x)\,dx\right) dr. \end{aligned} \tag{11.3}$$

As a consequence, for any $s > 0$, using the changes of variables $\rho = r/s$ and $y = x/s$, and making use of (11.2), we see that

$$\begin{aligned} E(s\tau_2) - E(s\tau_1) &= \int_{s\tau_1}^{s\tau_2} \left(\frac{1}{r^2}\int_{\partial B_r} Q(u,x)\,dx\right) dr \\ &= \int_{\tau_1}^{\tau_2} \left(\frac{1}{\rho^2}\int_{\partial B_\rho} Q(u,sy)\,dy\right) d\rho \\ &= \int_{\tau_1}^{\tau_2} \left(\frac{1}{\rho^2}\int_{\partial B_\rho} Q(u_s,y)\,dy\right) d\rho. \end{aligned} \tag{11.4}$$

On the other hand, by Theorem 1.13, we know that E is monotone and bounded, and therefore the limit as $\vartheta \to 0^+$ of $E(\vartheta)$ exists and it is finite. Consequently, we have that

$$E(s\tau_2) - E(s\tau_1) \to 0 \quad \text{as } s \to 0.$$

Hence, recalling (11.4), we conclude that

$$\int_{\tau_1}^{\tau_2} \left(\frac{1}{\rho^2}\int_{\partial B_\rho} Q(u_s,y)\,dy\right) d\rho \to 0 \quad \text{as } s \to 0. \tag{11.5}$$

Also, by compactness (ensured here, if u is a minimizer, by (1.24), which in turns allows us to exploit Corollary A.2, and, if u is a one-phase minimizer by the assumption that $u \in C^{1,1}(\Omega)$), we have that u_s converges to some u_0, up to a subsequence. Therefore, by (11.5),

$$\int_{\tau_1}^{\tau_2} \left(\frac{1}{\rho^2}\int_{\partial B_\rho} Q(u_0,y)\,dy\right) d\rho = 0$$

for all $\tau_2 > \tau_1 > 0$. Thus, since $Q \geqslant 0$, due to (11.1), it follows that $Q(u_0, y) = 0$. Consequently, by (11.3), we have that the function E relative to the minimizer u_0 is identically constant. Therefore, in view of the last claim in Theorem 1.13, it follows that u_0 is a homogeneous function of degree two. □

12. Regularity of the free boundary in two dimensions: explicit computations, classification results in $2D$, and proof of Theorem 1.15

In this section we study the regularity of free boundary of minimizers in dimension 2. Some of the results presented rely on direct calculations, while others are obtained by the monotone quantity E that has been analyzed in Theorems 1.13 and 1.14. In this setting, we have the following classification result for one-phase minimizers:

Theorem 12.1. *Let $u \in C^1(\mathbb{R}^n)$ be a one-phase local minimizer in any ball of $\mathbb{R}^n$, with $0 \in \partial_{sing}\{u > 0\}$. Let $u = r^2 g(\theta)$, where (r, θ) denotes the polar coordinates. Then, the following dichotomy holds:*

- *either u is a homogeneous polynomial of degree two,*
- *or, up to a rotation,*
$$u(x) = a\,(x_1^+)^2,$$
for some $a > 0$.

Proof. A direct computation shows that

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{1}{r^2}\,\Delta_{\mathbb{S}^1} u = 2g + 2g + g'' = g'' + 4g. \tag{12.1}$$

Accordingly, by Lemma 4.1, we have that, in the positivity set of u, we have

$$r^2\Delta^2 u = \frac{d^2}{d\theta^2}(g'' + 4g) = 0.$$

From this, we deduce that

$$g''(\theta) + 4g(\theta) = c_1\theta + c_2, \qquad \text{for all } \theta \in \{g \neq 0\}, \tag{12.2}$$

for some constants c_1 and c_2. We notice that (12.2) has explicit solution

$$\begin{aligned} g(\theta) &= \frac{c_1\theta}{4} + \frac{c_2}{4} + c_3\cos(2\theta) + c_4\sin(2\theta) \\ &= \frac{c_1\theta}{4} + \frac{c_2}{4} + c_3(\cos^2\theta - \sin^2\theta) + 2c_4\sin\theta\cos\theta, \end{aligned} \tag{12.3}$$

for some constants c_3 and c_4.

Since 0 is a free boundary point for u, we have that g cannot vanish identically. Hence, we distinguish some cases, depending on the number of zeros of g. First of all, we consider the cases in which either $g > 0$ for all $\theta \in [0, 2\pi)$ or g vanishes only at one point. Then, in this case the free boundary is contained in a ray and, up to a rotation, we can assume that $g(\theta) > 0$ for all $\theta \in (0, 2\pi)$ and so (12.3) is valid for all $\theta \in (0, 2\pi)$. The periodicity of g then implies that

$$0 = \lim_{\theta\to 0^+} g(\theta) - \lim_{\theta\to 2\pi^-} g(\theta) = -\frac{c_1\pi}{2},$$

and so $c_1 = 0$. As a consequence, by (12.3),

$$u(r,\theta) = \frac{c_2 r^2}{4} + c_3 r^2(\cos^2\theta - \sin^2\theta) + 2r^2 c_4\sin\theta\cos\theta = \frac{c_2(x_1^2 + x_2^2)}{4} + c_3(x_1^2 - x_2^2) + 2c_4 x_1 x_2,$$

which is a homogeneous polynomial of degree two, thus proving the desired claim in this case.

Now we suppose that g vanishes at least at two points, say, up to a rotation, θ_0 and $-\theta_0$, for some $\theta_0 \in (0, \pi)$, that is

$$\begin{gathered} g(\theta) > 0 \text{ for all } \theta \in (-\theta_0, \theta_0), \\ \text{and } g(\theta_0) = g(-\theta_0) = 0. \end{gathered} \tag{12.4}$$

Then, by (12.3),

$$0 = g(\pm\theta_0) = \pm\frac{c_1\theta_0}{4} + \frac{c_2}{4} + c_3\cos(2\theta_0) \pm c_4\sin(2\theta_0). \tag{12.5}$$

By the assumptions that $u \in C^1(\mathbb{R}^n)$ and $g \geqslant 0$, we also know that

$$0 = g'(\pm\theta_0) = \frac{c_1}{4} \mp 2c_3\sin(2\theta_0) + 2c_4\cos(2\theta_0). \tag{12.6}$$

Then, we obtain from (12.5) and (12.6) the system

$$\begin{cases} \dfrac{c_1\theta_0}{4} + c_4\sin(2\theta_0) = 0, \\ \dfrac{c_2}{4} + c_3\cos(2\theta_0) = 0, \\ c_3\sin(2\theta_0) = 0, \\ \dfrac{c_1}{4} + 2c_4\cos(2\theta_0) = 0. \end{cases} \tag{12.7}$$

Now, if

$$\theta_0 \neq \pi/2, \tag{12.8}$$

from (12.7) we have that necessarily $c_3 = 0$, and accordingly

$$\begin{cases} \dfrac{c_1\theta_0}{4} + c_4\sin(2\theta_0) = 0, \\ \dfrac{c_2}{4} = 0, \\ \dfrac{c_1}{4} + 2c_4\cos(2\theta_0) = 0. \end{cases}$$

This implies that $c_2 = 0$, and so (12.3) becomes

$$g(\theta) = \frac{c_1\theta}{4} + c_4\sin(2\theta).$$

In particular $g(0) = 0$, which is in contradiction with (12.4).

This says that the case in (12.8) must be ruled out, and thus $\theta_0 = \pi/2$ (and the positivity sets of u are either one or two halfplanes). In this way, the system in (12.7) reduces to

$$\begin{cases} \dfrac{c_1\pi}{8} = 0, \\ \dfrac{c_2}{4} - c_3 = 0, \\ \dfrac{c_1}{4} - 2c_4 = 0, \end{cases}$$

which leads to $c_1 = c_4 = 0$ and $\frac{c_2}{4} = c_3$. Substituting these conditions into (12.3), we obtain that, for all $\theta \in (-\pi/2, \pi/2)$,

$$g(\theta) = c_3\big(1 + \cos(2\theta)\big) = c_3\big(1 + \cos^2\theta - \sin^2\theta\big),$$

and therefore, for all $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 > 0$,

$$u(x) = 2c_3 x_1^2.$$

This gives that either u is a homogeneous polynomial of degree two, or $u(x) = a\,(x_1^+)^2$ for some $a > 0$, or

$$u(x) = \begin{cases} ax_1^2 & \text{if } x_1 \geqslant 0, \\ bx_1^2 & \text{if } x_1 < 0, \end{cases}$$

with a, $b \in (0, +\infty)$ and

$$a \neq b. \tag{12.9}$$

To complete the proof of the desired result, we need to exclude this case. To this end, we observe that

$$\begin{aligned} & \big(|\Delta u(0^+,1)|^2 + 1\big) - 2\big(\Delta u(0^+,1)u_{11}(0^+,1) - u_1(0^+,1)\Delta u(0^+,1)\big) \\ = \;& \big((2a)^2 + 1\big) - 2\big((2a)^2 + 0\big) \\ = \;& 1 - 4a^2, \end{aligned}$$

and similarly

$$\big(|\Delta u(0^-,1)|^2+1\big)-2\big(\Delta u(0^-,1)u_{11}(0^-,1)-u_1(0^-,1)\Delta u(0^-,1)\big) \quad = \quad 1-4b^2.$$

These identities and the free boundary condition (1.9) computed at the point $(0,1)$, where according to the definition in (1.6) we have $\lambda^{(1)}=\lambda^{(2)}=1$, lead to

$$1-4a^2=1-4b^2,$$

which gives that $a^2=b^2$ and thus $a=b$. This is in contradiction with (12.9), and the desired result is established. □

With this, we are now in the position of completing the proof of Theorem 1.15.

Proof of Theorem 1.15. Let E be as in Theorem 1.13, and let[2]

$$E(0):=\lim_{\rho\to 0^+}E(\rho). \tag{12.10}$$

Let $\bar x\in\partial\{u>0\}$. Suppose that $u_{0,\bar x}$ is a blow-up of u at $\bar x$. Notice that $u_{0,\bar x}$ cannot be identically equal to zero, due to (1.26). Then by Theorem 12.1 we know that, after some rotation of coordinates,

$$\begin{gathered} u_{0,\bar x} \text{ must be one of the following functions:}\\ \frac{a_1(x_1-\bar x_1)^2+a_2(x_2-\bar x_2)^2}{2},\quad \frac{a(x_1-\bar x_1)^2}{2},\quad \frac{a((x_1-\bar x_1)^+)^2}{2}, \end{gathered} \tag{12.11}$$

with a_1, a_2, $a>0$ (say, possibly depending on $\bar x$, though the free boundary conditions in Theorem 1.3 have to be fulfilled).

In particular, from (12.11), we know that

$$\Delta u \text{ is constant in the positivity cone of } u. \tag{12.12}$$

Now, from (1.25), we know that, if

$$u_{k,\bar x}(x):=\frac{u(\bar x+\rho_k x)}{\rho_k^2}, \tag{12.13}$$

with $\rho_k\to 0^+$, then, up to a subsequence,

$$u_{k,\bar x}\to u_{0,\bar x} \text{ in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n), \tag{12.14}$$

as $k\to+\infty$, for any $\alpha\in(0,1)$.

We claim that

$$u_{0,0} \text{ must necessarily be } \frac{a(x_1^+)^2}{2}, \tag{12.15}$$

namely the first and the second possibilities in (12.11) are excluded at the origin. To prove (12.15), we argue by contradiction. If not, by (12.14) and (12.11), necessarily

$$\frac{u(\rho_k x)}{\rho_k^2}=u_{k,0}(x)\to\left\{\begin{matrix}\text{either } \dfrac{a_1x_1^2+a_2x_2^2}{2},\\[2ex] \text{or } \dfrac{ax_1^2}{2}\end{matrix}\right\}=:u_{0,0}(x)$$

in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore, using the change of variable $y:=\rho_k x$,

$$\begin{aligned}\lim_{k\to+\infty}\frac{|B_{\rho_k}\cap\{u>0\}|}{|B_{\rho_k}|}&=\lim_{k\to+\infty}\frac{1}{|B_{\rho_k}|}\int_{B_{\rho_k}\cap\{u>0\}}dx\\ &=\lim_{k\to+\infty}\frac{1}{|B_1|}\int_{B_1\cap\{u_{k,0}>0\}}dy=\frac{1}{|B_1|}\int_{B_1\cap\{u_{0,0}>0\}}dy=1.\end{aligned}$$

This is a contradiction with (1.27), and so (12.15) is proved.

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[2] We observe that the limit in (12.10) exist, due to the monotonicity of E, recall Theorem 1.13.

We let $E_{k,\bar x}$ be the monotone function in (1.22) for $u_{k,\bar x}$ (while $E_{\bar x}$ denotes the same type of function for u centered at the point $\bar x$). Let also $E_{0,\bar x}$ be the monotone function in (1.22) for $u_{0,\bar x}$. In view of (12.14), we have that

$$E_{0,\bar x}(r)=\lim_{k\to+\infty}E_{k,\bar x}(r). \tag{12.16}$$

We remark that (1.22) is compatible with the blow-up scaling, namely

$$E_{k,\bar x}(r)=E_{\bar x}(\rho_k r).$$

As a consequence, by (12.10) and (12.16),

$$E_{0,\bar x}(r)=\lim_{k\to+\infty}E_{\bar x}(\rho_k r)=E_{\bar x}(0). \tag{12.17}$$

We now classify the free boundary points according to the monotone function induced by their blow-up limits. For this, we introduce the following notation: recalling (12.11), we say that $\bar x$ is Type-1 if, up to a rotation,

$$u_{0,\bar x}(x)=\frac{a_1(x_1-\bar x_1)^2+a_2(x_2-\bar x_2)^2}{2}.$$

Similarly, we say that $\bar x$ is Type-2 if

$$u_{0,\bar x}(x)=\frac{a(x_1-\bar x_1)^2}{2},$$

and Type-3 if

$$u_{0,\bar x}(x)=\frac{a((x_1-\bar x_1)^+)^2}{2}.$$

In this notation, (12.15) says that the origin is Type-3.

Now, in light of (1.22) and Lemma 10.3 (which can be utilized here thanks to (12.12)), we have that

$$E_{0,\bar x}(r)=\begin{cases}\pi/4, & \text{if } \bar x \text{ is either Type-1 or Type-2,}\\ \pi/8, & \text{if } \bar x \text{ is Type-3.}\end{cases} \tag{12.18}$$

In particular, the monotone function E is minimized for Type-3 free boundary points.

Moreover, we have the following semicontinuity property: if $x_j\in\partial\{u>0\}$ and $x_j\to x_0$ as $j\to+\infty$, then

$$\limsup_{j\to+\infty}E_{x_j}(0)\leqslant E_{x_0}(0). \tag{12.19}$$

Indeed, by the monotonicity of E in Theorem 1.13 and (1.22), for any $r\in(0,1)$ we have that

$$\limsup_{j\to+\infty}E_{x_j}(0)\leqslant\limsup_{j\to+\infty}E_{x_j}(r)=E_{x_0}(r).$$

Then, we take the limit as $r\to0^+$ and we obtain (12.19), as desired.

Now we claim that there exists $r_0>0$ such that

$$\text{for any } \bar x\in\partial\{u>0\}\cap B_{r_0} \text{ we have that } E_{\bar x}(0)=E_0(0). \tag{12.20}$$

In other words, in B_{r_0} all free boundary points must be of Type-3. To prove this we argue by contradiction: if not there exists a sequence of points $\bar x_j\in\partial\{u>0\}$ such that $\bar x_j\to0$ as $j\to+\infty$ and

$$E_{\bar x_j}(0)\neq E_0(0). \tag{12.21}$$

From (12.11), (12.15), (12.17), (12.18) and (12.21), we deduce that

$$\left\{\frac{\pi}{8},\frac{\pi}{4}\right\}\ni E_{0,\bar x_j}(r)=E_{\bar x_j}(0)\neq E_0(0)=E_{0,0}(r)=\frac{\pi}{8},$$

and accordingly

$$E_{\bar x_j}(0)=E_{0,\bar x_j}(r)=\frac{\pi}{4}>\frac{\pi}{8}=E_{0,0}(r)=E_0(0).$$

This gives that

$$\lim_{j\to+\infty}E_{\bar x_j}(0)=\frac{\pi}{4}>\frac{\pi}{8}=E_0(0),$$

which is in contradiction with (12.19), and so the proof of (12.20) is complete.

Then, by (12.18) and (12.20), it follows that if $\bar{x} \in \partial\{u > 0\} \cap B_{r_0}$ then $\bar{x}$ must necessarily be Type-3, i.e., up to rotations, $u_{0,\bar{x}}(x) = \frac{a((x_1-\bar{x}_1)^+)^2}{2}$, which is the desired result. $\square$

Appendix A. Decay estimate for D^2u

Here we provide some decay estimates for the gradient and the Hessian of a local minimizer of the functional J in (1.1).

Lemma A.1. *Let $n \geqslant 2$, u be a local minimizer for the functional J defined in (1.1), and $x_0 \in \partial\{u > 0\}$.*

Then, there exist $R_0 > 0$ and $C > 0$, depending only on n and $\|u\|_{W^{2,2}(\Omega)}$, such that

$$\frac{1}{R^{n+2}}\int_{B_R(x_0)} |\nabla u|^2 + \frac{1}{R^n}\int_{B_R(x_0)} |D^2u|^2 \leqslant \frac{C}{R^{n+4}}\int_{B_{4R}(x_0)} (u-m)^2 + \frac{\widehat{C}}{R^{n+2}}\int_{B_{4R}} (u-m),$$

for any $R < R_0$, where

$$m := \min_{B_{4R}(x_0)} u \tag{A.1}$$

and $\widehat{C}$ is the constant appearing in Corollary 4.2.

Proof. Without loss of generality we suppose that $x_0 = 0$. Recalling Lemma 4.1, we have that, for any $\phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, with $\phi \geqslant 0$, it holds that

$$0 \geqslant \int_\Omega \Delta u\, \Delta\phi. \tag{A.2}$$

Now, let $\phi \in C_0^\infty(\Omega)$, with $\phi \geqslant 0$, and define $\phi_\varepsilon := \phi * \rho_\varepsilon$, where $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$, for any $x \in \mathbb{R}^n$, is a mollifying kernel, for any $\varepsilon \in (0,1)$. We also set $u_\varepsilon := u * \rho_\varepsilon$. Then, if $\mathrm{dist}(\mathrm{supp}\phi, \partial\Omega) \gg \varepsilon$, we can use (A.2) and make an integration by parts twice to obtain that

$$\begin{aligned}
0 \geqslant \int_\Omega \Delta u\, \Delta\phi_\varepsilon &= \int_\Omega \Delta u (\Delta\phi) * \rho_\varepsilon \\
&= \int_\Omega \Delta u(x) \left(\int_\Omega \rho_\varepsilon(x-y)\Delta\phi(y)dy\right) dx \\
&= \int_\Omega \Delta\phi(y) \left(\int_\Omega \rho_\varepsilon(x-y)\Delta u(x)dx\right) dy \\
&= \int_\Omega \Delta\phi\, \Delta u_\varepsilon \\
&= \sum_{i,j=1}^n \int_\Omega \phi_{ii}(u_\varepsilon)_{jj} \\
&= \sum_{i,j=1}^n \int_{\partial\Omega} \phi_i(u_\varepsilon)_{jj}\nu^i - \sum_{i,j=1}^n \int_\Omega \phi_i(u_\varepsilon)_{ijj} \\
&= \sum_{i,j=1}^n \int_{\partial\Omega} \phi_i(u_\varepsilon)_{jj}\nu^i - \sum_{i,j=1}^n \int_{\partial\Omega} \phi_i(u_\varepsilon)_{ij}\nu^j + \sum_{i,j=1}^n \int_\Omega \phi_{ij}(u_\varepsilon)_{ij} \\
&= \sum_{i,j=1}^n \int_\Omega \phi_{ij}(u_\varepsilon)_{ij}.
\end{aligned} \tag{A.3}$$

Moreover, we observe that

$$\lim_{\varepsilon\to 0}\int_\Omega \phi_{ij}(u_\varepsilon)_{ij} = \int_\Omega \phi_{ij}u_{ij}.$$

From this and (A.3), we have that

$$\sum_{i,j=1}^n \int_\Omega \phi_{ij}u_{ij} \leqslant 0. \tag{A.4}$$

Now, we choose $\phi := (u-m)\eta^2$, where m is as in (A.1), and η is a standard cut-off function supported in $B_{2R} \Subset \Omega$, such that $\eta = 1$ in B_R and $\eta = 0$ outside B_{2R}. Therefore, we see that $\phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $\phi \geqslant 0$. With this choice,

$$\phi_{ij} = u_{ij}\eta^2 + 2u_i\eta_j\eta + 2u_j\eta_i\eta + (u-m)(\eta^2)_{ij}.$$

If we plug this into (A.4), we have that

$$\sum_{i,j=1}^n \int_\Omega \Big(u_{ij}\eta^2 + 4u_i\eta_j\eta + (u-m)(\eta^2)_{ij} \Big) u_{ij} \leqslant 0.$$

That is, rearranging the terms and integrating by parts,

$$\begin{aligned} \sum_{i,j=1}^n \int_\Omega u_{ij}^2\eta^2 \leqslant & -\sum_{i,j=1}^n \int_\Omega \Big(4u_{ij}u_i\eta_j\eta + (u-m)u_{ij}(\eta^2)_{ij} \Big) \\ = & -\sum_{i,j=1}^n \int_\Omega 4(u_{ij}\eta)u_i\eta_j + \sum_{i,j=1}^n \int_\Omega \Big((u-m)u_i(\eta^2)_{ijj} + u_iu_j(\eta^2)_{ij} \Big) \\ \leqslant & \, 2\delta \sum_{i,j=1}^n \int_\Omega u_{ij}^2\eta^2 + \frac{8}{\delta}\sum_{i,j=1}^n \int_\Omega u_i^2\eta_j^2 + \sum_{i,j=1}^n \int_\Omega \Big((u-m)u_i(\eta^2)_{ijj} + u_iu_j(\eta^2)_{ij} \Big), \end{aligned} \tag{A.5}$$

where the last line follows from a suitable application of the Hölder inequality, for some $\delta > 0$.

Now, by direct computations we have

$$\begin{aligned} (\eta^2)_{ij} &= 2\eta_i\eta_j + 2\eta\eta_{ij} \\ \text{and} \quad (\eta^2)_{ijj} &= 2\eta_{ij}\eta_j + 2\eta_i\eta_{jj} + 2\eta_j\eta_{ij} + 2\eta\eta_{ijj}, \end{aligned}$$

and therefore

$$\big|(\eta^2)_{ij}\big| \leqslant \frac{C}{R^2} \quad \text{and} \quad \big|(\eta^2)_{ijj}\big| \leqslant \frac{C}{R^3},$$

for some $C > 0$.

As a consequence, plugging this information into (A.5) and using the Hölder inequality, we obtain that

$$\begin{aligned} (1-2\delta)\sum_{i,j=1}^n \int_\Omega u_{ij}^2\eta^2 &\leqslant \frac{8}{\delta}\sum_{i,j=1}^n \int_\Omega u_i^2\eta_j^2 + \sum_{i,j=1}^n \int_\Omega \Big((u-m)u_i(\eta^2)_{ijj} + u_iu_j(\eta^2)_{ij} \Big) \\ &\leqslant \frac{C}{\delta R^2}\int_{B_{2R}} |\nabla u|^2 + \frac{C}{R^3}\int_{B_{2R}} (u-m)|\nabla u| + \frac{C}{R^2}\int_{B_{2R}} |\nabla u|^2 \\ &\leqslant \left(1+\frac{1}{\delta}\right)\frac{C}{R^2}\int_{B_{2R}} |\nabla u|^2 + \frac{C}{R^4}\int_{B_{2R}} (u-m)^2, \end{aligned} \tag{A.6}$$

up to renaming C. Since $\Delta u \geqslant -\hat{C}$ (recall Corollary 4.2), then from the Caccioppoli inequality (see e.g. (7.7)) we get that

$$\int_{B_{2R}} |\nabla u|^2 \leqslant \frac{C}{R^2}\int_{B_{4R}} (u-m)^2 + C\int_{B_{4R}} (u-m),$$

which implies that

$$\frac{1}{R^{n+2}}\int_{B_{2R}} |\nabla u|^2 \leqslant \frac{C}{R^{n+4}}\int_{B_{4R}} (u-m)^2 + \frac{C}{R^{n+2}}\int_{B_{4R}} (u-m). \tag{A.7}$$

Moreover, from (A.6) and (A.7), we conclude that

$$\frac{1-2\delta}{R^n}\sum_{i,j=1}^n \int_{B_R} u_{ij}^2 \leqslant \frac{1-2\delta}{R^n}\sum_{i,j=1}^n \int_\Omega u_{ij}^2\eta^2 \leqslant \frac{C}{R^{n+4}}\int_{B_{4R}} (u-m)^2 + \frac{C}{R^{n+2}}\int_{B_{4R}} (u-m)$$

up to renaming $C > 0$. Putting together this and (A.7), we obtain the desired estimate. ∎

Corollary A.2. *Let $n \geqslant 2$, $\delta > 0$ and u be a local minimizer of the functional J defined in (1.1). Let $x_0 \in \partial\{u > 0\}$ such that $\nabla u(x_0) = 0$ and $\partial\{u > 0\}$ is not δ-rank-2 flat at x_0 at any level $r > 0$ in the sense of Definition 1.6.*

Then, there exist $R_0 > 0$ and $C > 0$, depending only on n and $\|u\|_{W^{2,2}(\Omega)}$, such that

$$\frac{1}{R^{n+2}}\int_{B_R(x_0)} |\nabla u|^2 + \frac{1}{R^n}\int_{B_R(x_0)} |D^2 u|^2 \leqslant C,$$

for any $R < R_0$.

Proof. The desired estimate follows from Lemma A.1 and the quadratic growth of u, as given by Theorem 1.7. □

Appendix B. A remark on the one-phase problem

Here we show that the one-phase problem, as presented in Definition 1.2, and the analysis of the minimizers which happen to be nonnegative are structurally very different questions. Indeed, while a "typical" one-phase minimizer exhibits nontrivial open regions in which it vanishes, the free minimizers that are nonnegative do not show the same phenomena. As a prototype result for this, we point out the following observation:

Proposition B.1. *Suppose that $0 \in \Omega$, $u \in C^{1,1}(\Omega)$ is such that $u > 0$ in $\Omega \cap \{x_n > 0\}$ and $u = 0$ in $\Omega \cap \{x_n \leqslant 0\}$. Then, u cannot be a local minimizer for the functional J in Ω in the class of admissible functions $\mathcal{A}$ given in (1.2).*

Proof. Without loss of generality, we can assume that $B_2 \Subset \Omega$. Let $\varphi \in C_0^\infty(B_2, [0, 1])$ be such that $\varphi = 1$ in B_1. Let also $\varepsilon \in (0, 1)$ and $u_\varepsilon := u - \varepsilon\varphi$.

We observe that the regularity of u and the fact that $u(x', 0) = 0 \leqslant u(y)$ for any x' such that $(x', 0) \in B_2$ and any $y \in B_2$ give that, for every $x = (x', x_n) \in B_1$,

$$u(x) \leqslant K\, x_n^2,$$

for some $K > 0$. Consequently, for every $x \in B_1$ with $|x_n| < \sqrt{\varepsilon/K}$ we have that

$$u_\varepsilon(x) \leqslant K\, x_n^2 - \varepsilon < 0.$$

This gives that for any $x \in (-1/n, 1/n)^{n-1} \times (0, \sqrt{\varepsilon/K}) =: Q_\varepsilon$, we have that

$$u_\varepsilon(x) < 0 < u(x),$$

as long as $\varepsilon > 0$ is sufficiently small.

Notice also that $u_\varepsilon \leqslant u$ and so $\{u_\varepsilon > 0\} \subseteq \{u > 0\}$. Accordingly, computing the energy functional in B_2,

$$\begin{aligned} J[u_\varepsilon] - J[u] &= \int_{B_2} \left(|\Delta u_\varepsilon|^2 - |\Delta u|^2\right) + |B_2 \cap \{u_\varepsilon > 0\}| - |B_2 \cap \{u > 0\}| \\ &= \int_{B_2} \left(|\Delta u - \varepsilon\Delta\varphi|^2 - |\Delta u|^2\right) - |B_2 \cap \{u_\varepsilon \leqslant 0 < u\}| \\ &\leqslant \int_{B_2} \left(\varepsilon^2|\Delta\varphi|^2 - 2\varepsilon\Delta u\Delta\varphi\right) - |Q_\varepsilon| \\ &\leqslant C\varepsilon - \left(\frac{2}{n}\right)^{n-1}\sqrt{\frac{\varepsilon}{K}} \\ &< 0 \end{aligned}$$

provided that ε is small enough. □

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